Applications of stochastic control to real options and to liquidity risk model.

Vathana Ly Vath

Université Paris VII - Denis Diderot
UFR de Mathématiques

Année 2006

Thèse

pour obtenir le titre de

Docteur de l'Université Paris 7

Spécialité : Mathématiques Appliquées

présentée par

LY VATH Vathana

Quelques Applications du Contrôle Stochastique aux Options Réelles et au Risque de Liquidité

Directeur de thèse

Pr. PHAM Huyên

Soutenue publiquement le 4 Décembre 2006, devant le jury composé de :

Pr. JEANBLANC Monique, Université d'Evry Val d'Essonne
Pr. LAMBERTON Damien, Université de Marne-La-Vallée
Pr. PHAM Huyên, Université Paris 7
Dr. SULEM Agnès, INRIA
Pr. TOUZI Nizar, Ecole Polytechnique

au vu des rapports de :

Pr. TOUZI Nizar, Ecole Polytechnique
Pr. ZERVOS Mihail, London School of Economics

A ma famille

Mes premiers remerciements vont naturellement à mon directeur de thèse, Huyên Pham, sans qui cette thèse n'aurait jamais pu voir le jour. Je le remercie de m'avoir initié à la recherche dans le domaine des mathématiques financières et du contrôle stochastique. Je le remercie également pour ses nombreux conseils, son soutien constant, ainsi que pour la rigueur mathématique qu'il m'a apportée. Je lui suis particulièrement reconnaissant pour sa disponibilité tout au long de ces trois dernières années.

Je suis très reconnaissant envers Nizar Touzi et Mihail Zervos pour leur intérêt dans mes travaux de thèse, en acceptant de la rapporter. Je remercie également Monique Jeanblanc, Agnès Sulem et Damien Lamberton d'avoir accepté de participer au jury.

Mes remerciements vont également à tous les enseignants du DEA de Statistique et Modèles Aléatoires en Finance, en particulier, Laure Elie.

Je suis également reconnaissant envers Mohamed Mnif pour son aide et sa collaboration sur le modèle de risque de liquidité, en particulier, sur la partie numérique, et envers Stéphane Villeneuve pour sa collaboration sur le problème couplé de contrôle singulier et de changement de régime.

J'exprime également ma reconnaissance envers Michèle Wasse pour toutes ses aides administratives pendant ces trois années passées à Chevaleret. Sans elle, le début de l'année scolaire de chaque étudiant ressemblerait certainement à un parcours du combattant.

Mes pensées vont enfin aux amis thésards du Chevaleret, en particulier, ceux du bureau 5B01, pour la bonne ambiance et surtout pour leurs amitiés. Je tiens à remercier, en particulier, Benjamin Bruder, Christophe Chesneau, Emel et Kadir Erdogan, Fernand Malonga, Frederic Guilloux, Katia Meziani, Nuray Caliskan, Thomas Willer, et Tu Nguyen.

TABLE DES MATIÈRES

INTRODUCTION GÉNÉRALE

0.1 Introduction et motivations

L'évaluation des actifs et la gestion de portefeuille, deux problèmes fondamentaux de finance, ont subi de nombreux bouleversements pendant ces dernières décennies. Si le calcul actuariel était pratiquement le seul outil mathématique utilisé par les financiers jusqu'au début des années 70, le développement des mathématiques financières a totalement transformé le monde de la finance.

Les bouleversements s'opèrent d'abord dans la diversification et la prolifération des produits financiers (produits dérivés, structurés...), ensuite, dans la sophistication de ces produits permettant une fiabilité accrue dans leur évaluation, et enfin dans le développement des théories de la gestion de portefeuille.

Les impacts sur les activités financières et économiques sont profonds. Ils entraînent une multiplication du nombre d'intervenants sur le marché financier (gérants de fonds, entreprises industrielles et commerciales...) augmentant ainsi la liquidité du marché, une satisfaction accrue des besoins de ces derniers, et surtout une meilleure gestion des risques. Pour les financiers, une meilleure gestion des risques signifie une amélioration dans la couverture des positions risquées limitant ainsi des pertes éventuelles. Dans le monde industriel et économique, elle permet surtout une meilleure planification budgétaire et encourage les investissements pour l'avenir. En résumé, ces bouleversements contribuent significativement aux développements économiques ces dernières décennies.

Un des principaux moteurs de ces innovations est, sans aucun doute, le développement de la théorie de l'optimisation et du contrôle stochastique. Développé dans les années 70, le contrôle stochastique a reçu de nouvelles attentions de la communauté des mathématiques financières. La recherche se tourne désormais vers des nouveaux champs d'applications de cette théorie qui s'étendent à de multiples domaines, en particulier, en économie et en industrie. De nombreux problèmes laissés en suspens par les industriels, économistes et financiers trouvent ainsi des éléments de réponse dans la théorie du contrôle stochastique.

Le contrôle stochastique est l'étude des systèmes dynamiques soumis aux perturbations aléatoires qui peuvent être controlées dans le but d'optimiser certains critères de performance tels que la maximisation des profits et de l'utilité de la valeur liquidative terminale.

Cette thèse présente quelques applications du contrôle stochastique, en particulier, au risque de liquidité et aux options réelles, deux thèmes parmi les plus étudiés actuellement dans la littérature économique et financière. Elle s'organise de la manière suivante.

Dans la première partie, l'étude porte sur la sélection du portefeuille optimal sous un modèle de risque de liquidité. Ici, on entend par liquidité, la liquidité du marché qui correspond à la possibilité pour un investisseur d'effectuer une transaction au prix affiché et pour un volume important sans affecter le cours du titre. Elle est d'autant plus forte que le nombre de titres admis sur le marché est important et que la fréquence des transactions est élevée. Dans les modèles classiques du marché financier, on fait l'hypothèse d'un marché financier parfaitement liquide, ce qui ne correspond guère à la réalité du marché. En effet, le marché de la plupart des actifs est peu liquide et représente donc un risque pour les investisseurs concernés. Ces derniers affectent généralement une décote pour de tels actifs. Dans cette partie, on étudie un problème de sélection de portefeuille optimal d'un investisseur sous un modèle de risque de liquidité. Le critère consiste à maximiser l'espérance de l'utilité de la valeur terminale de liquidation du portefeuille sous certaines contraintes de solvabilité. Des méthodes numériques d'itération d'une stratégie optimale sont également traitées dans le chapitre 2 de la première partie.

Dans la deuxième partie de la thèse, seront traités deux problèmes d'optimisation stochastique, assimilables aux options réelles. Par analogie avec l'option du financier, on parle d'option réelle pour caractériser la position d'un industriel qui bénéficie d'une certaine flexibilité dans la gestion de l'entreprise, par exemple, un projet d'investissement. Il est, en effet, possible de limiter ou d'accroître le niveau d'investissement compte tenu de l'évolution des perspectives économiques et de rentabilité, tout comme un financier peut exercer ou non son option sur un sous-jacent. Cette flexibilité détient une valeur qui est tout simplement la valeur de l'option réelle. Le premier problème, dans le chapitre 3, concerne la résolution d'un problème d'optimisation de changement de régime à deux états. Le deuxième problème, dans le chapitre 4, traite un problème couplé de contrôle singulier et de changement de régime dans le cadre de la politique de dividende avec investissement réversible.

Enfin, dans la troisième et dernière partie, on étudie l'existence d'un équilibre dans un marché compétitif sous asymétrie d'information.

Dans la résolution des problèmes des deux premières parties, et dans une moindre mesure, de la dernière partie, des techniques de contrôle stochastique seront utilisées. L'approche typique consiste à exprimer le principe de la programmation dynamique lié à chaque problématique afin d'obtenir une caractérisation par EDP (Equations aux Dérivées Partielles) des fonctions de valeur. Par cette approche, on est capable de montrer dans le problème de risque de liquidité et les deux options réelles que les fonctions de valeur correspondantes sont l'unique solution au système d'inégalités variationnelles d'Hamilton-Jacobi-Bellman associé. Autrement dit, les fonctions de valeur satisfont à fois les propriétés de viscosité et le principe de comparaison.

Dans chaque problème des deux premières parties, on peut obtenir les solutions, en particulier le contrôle optimal correspondant, soit d'une manière explicite (chapitre 3 et 4), soit par une méthode itérative (chapitre 1 et 2).

Dans la suite de cette introduction, nous allons exposer la problématique de chaque chapitre ainsi que les résultats importants obtenus.

0.2 Un modèle de risque de liquidité

0.2.1 Aspects théoriques

Dans l'article de référence de Merton [53], l'auteur a examiné un problème en temps continu de consommation-investissement d'un individu. Dans une optique de gestion de portefeuille, il cherche à déterminer la proportion optimale de richesse que l'investisseur doit détenir pour chaque actif du marché en fonction de son prix. En utilisant le critère de maximisation d'utilité et des techniques de contrôle stochastique, il a obtenu une formule explicite de la fonction de valeur et la stratégie optimale correspondante. Comme dans tous les modèles classiques en mathématiques financières, il considère une parfaite élasticité des actifs, en supposant que les transactions n'ont aucun impact sur le prix de l'actif.

Cependant, la littérature sur la microstructure du marché a montré théoriquement et empiriquement que les grosses transactions influencent significativement le prix de l'actif sous-jacent, démontrant ainsi l'existence du risque de liquidité. Si l'hypothèse d'un marché parfaitement liquide ne représente que peu d'importance pour les décisions d'allocation d'actifs sur le long terme, l'impact de prix dû au risque de liquidité influence significativement les décisions d'investissement des gros investisseurs focalisant sur un horizon de temps relativement court.

Dans la littérature actuelle, trois principales approches ont été suggérées pour formaliser cette notion de risque de liquidité. Dans les travaux de Back [3] et de Kyle [48], l'impact des stratégies de trading sur les prix est expliqué par la présence d'un agent initié. Dans la littérature sur la manipulation du marché, les prix sont considérés dépendants directement des stratégies de transaction. Dans [20], Cuoco et Cvitanic considèrent un modèle de diffusion pour les dynamiques de prix avec des coefficients dépendant de la stratégie des gros investisseurs, alors que Frey [30], Platen et Schweizer[58], Papanicolaou et Sircar [56], Bank et Baum [4], Cetin, Jarrow et Protter [14] développent un modèle en temps continu où les prix dépendent des stratégies via une fonction de réaction. Dans [16], Cetin, Soner et Touzi se placent dans le cadre du modèle développé par Cetin, Jarrow et Protter [14], afin d'étudier le problème de couverture des options, en particulier, le problème de sur-réplication, en présence de coût de liquidité. La troisième et dernière approche établit que le coût de transaction est également un facteur déterminant dans le comportement des investisseurs. Pour cela, on peut se référer aux travaux de Davis-Norman [22], Korn [47], Oksendal et Sulem [55], Vayanos [65] et de Lo, Mamayski et Wang [50] qui illustrent parfaitement l'influence des coûts de transaction sur la liquidité du marché et les prix.

Dans notre étude, on considère un modèle prenant en compte à la fois le coût de transaction et la manipulation du marché, deux phénomènes qui font simultanément partie de la réalité du marché financier. Inspiré des papiers récents de Subramanian et Jarrow [63] et de He et Mamaysky [38], notre modèle suppose l'existence d'un gros investisseur dont les transactions influencent les cours des actifs : un achat entraîne une hausse de prix, alors qu'une vente entraîne une baisse.

Comme dans l'article de Merton [53], on considère un marché comportant un actif sans-risque avec un taux d'intérêt constant $r > 0$ et un actif risqué gouverné par un brownien géométrique. L'objectif est d'obtenir la stratégie optimale, autrement dit, la proportion optimale de chaque actif, maximisant l'espérance de l'utilité de la valeur liquidative au temps terminal T sous la contrainte de solvabilité suivante : sa valeur liquidative à chaque instant doit être positive, $t \in [0,T]$, $L(Z_t) := L(X_t, Y_t, P_t) \;\geq\; 0$, où X, Y et P processus représentant respectivement la quantité de cash, le nombre cumulé d'actif risqué et son prix dans le portefeuille.

On considère, en particulier, un coût de transaction fixe, $k > 0$, et une fonction d'impact de prix exponentielle : lors d'une transaction de y actions de l'actif risqué, le prix de l'actif passe du prix pré-trade p à un prix post-trade $pe^{\lambda y}$, avec $\lambda > 0$, une constante positive donnée. Ainsi, quand un agent achète y parts de l'actif risqué, il doit payer $k + ype^{\lambda y}$. De même, une vente de y parts résulterait en une réception de $\;-k + ype^{-\lambda y}$.

Formulation du problème. L'hypothèse de coût de transaction fixe impose un modèle à transaction discrète. On modélise ainsi ce problème d'optimisation par une stratégie de contrôle impulsionnel $\alpha = (\tau_n, \xi_n)_{n\leq 1}$: $\tau_1 \leq ...\tau_n \leq ... < T$ représentent les temps d'intervention de l'investisseur, et ξ_n, le nombre d'actif risqué acheté ou vendu lors de ces interventions.

Le problème d'investissement. On étudie le problème de maximisation de l'espérance de l'utilité de la richesse liquidative terminale et considère la fonction de valeur suivante :

$$v(t,z) \quad = \quad \sup_{\alpha\in\mathcal{A}(t,z)} \mathbb{E}\left[U(L(Z_T))\right], \qquad (t,z) \in [0,T] \times \bar{\mathcal{S}}. \tag{0.2.1}$$

où $\mathcal{A}(t,z)$ représente l'ensemble des contrôles impulsionnels admissibles. Ce problème d'optimisation est associé par le principe de la programmation dynamique à l'inégalité quasi-variationnelle d'Hamilton-Jacobi-Bellman [7] suivante :

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v \;,\; v - \mathcal{H}v\right] \quad = \quad 0, \quad \text{sur} \quad [0,T) \times \mathcal{S}. \tag{0.2.2}$$

Résultats. L'objectif principal est d'obtenir une caractérisation rigoureuse de la fonction de valeur, et d'extraire, si possible, la stratégie optimale correspondante. Un recours aux notions de viscosité s'avère être un outil puissant pour la résolution de ce problème. Mais compte tenu de la non-linéarité de la fonction d'impact de prix et de la contrainte de solvabilité, plusieurs difficultés techniques apparaissent : la discontinuité de la fonction de valeur sur la frontière de solvabilité et à l'instant terminal T.

Pour montrer les propriétés de viscosité de la fonction de valeur, on utilise la notion de solution de viscosité sous contrainte introduite par Soner [62] et on considère seulement les solutions discontinues. En effet, la continuité de la fonction de valeur à l'intérieur de la région de solvabilité ne peut s'obtenir que d'une manière indirecte, autrement dit, après avoir prouvé le théorème de comparaison. Ce dernier s'obtient en utilisant les techniques développées par Ishii [40] et Barles [5].

Théorème. *La fonction de valeur v est continue sur $[0,T) \times \mathcal{S}$ et est l'unique solution de viscosité (sur $[0,T) \times \mathcal{S}$) sous contrainte (0.2.2) satisfaisant les conditions aux bords et au temps terminal et la condition de croissance :*

$$|v(t,z)| \quad \leq \quad K\left(1+\left(x+\frac{p}{\lambda}\right)\right)^{\gamma}, \quad \forall (t,z) \in [0,T) \times \mathcal{S} \tag{0.2.3}$$

pour un certain réel positif $K < \infty$.

0.2.2 Aspects numériques

Comme dans la plupart des problèmes de contrôle stochastique, il s'avère impossible d'obtenir explicitement l'expression de la fonction de valeur et la stratégie optimale correspondante. Pour résoudre ces problèmes, on se tourne alors vers la résolution numérique de l'inégalité quasi-variationnelle d'Hamilton-Jacobi-Bellman (IQVHJB) associée en faisant appel, généralement, aux méthodes des différences finies. L'algorithme de Howard, qui cherche à calculer d'une manière itérative la fonction de valeur et la stratégie optimale, est connu pour son efficacité dans la résolution de ces types d'équations. Dans [17], Chancelier, Oksendal et Sulem font appel à cet algorithme pour résoudre numériquement une IQVHJB de dimension 2 associée à un problème de consommation optimale pour un portefeuille avec coût de transaction fixe et proportionnel. Cependant, dans notre étude, la résolution numérique par l'algorithme de Howard n'est pas évidente compte tenu de la dimension de notre problème et surtout de la complexité de notre région de solvabilité.

Dans l'étude d'un problème de sélection de portefeuille optimal [47], Korn a présenté une suite de problèmes de temps d'arrêt optimaux et prouvé sa convergence vers la fonction de valeur initiale. Dans [17], les auteurs ont proposé une méthode itérative pour résoudre le problème de contrôle impulsionnel. Ils considèrent une fonction de valeur auxiliaire où le nombre de transactions est majoré par un nombre positif.

Dans ce chapitre, nous montrons que les deux méthodes itératives coïncident et que notre problème de contrôle impulsionnel se réduit à un problème itératif de problèmes d'arrêt optimaux. Avec un recours aux méthodes de Monte Carlo, nous donnons également un algorithme d'approximation numérique pour chacun de ces problèmes d'arrêt optimaux.

Convergence du schéma itératif. Nous introduisons les sous-ensembles de $\mathcal{A}(t,z)$: $\mathcal{A}_n(t,z) := \{\alpha = (\tau_k,\xi_k)_{k=0,\dots,n} \in \mathcal{A}(t,z)\}$, et considérons les fonctions de valeur, v_n, obtenues quand l'investisseur ne peut effectuer qu'au plus n interventions :

$$v_n(t,z) := \sup_{\alpha \in \mathcal{A}_n(t,z)} \mathbb{E}[U(L(Z_T))] \quad (t,z) \in [0,T] \times \overline{\mathcal{S}}. \tag{0.2.4}$$

Nous définissons également une suite itérative de problèmes d'arrêt optimaux :

$$\begin{cases} \varphi_{n+1}(t,z) & = & \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}\left[\mathcal{H}\varphi_n(\tau, Z_\tau^{0,t,z})\right], \\ \varphi_0(t,z) & = & v_0(t,z), \end{cases}$$

où $\mathcal{S}_{t,T}$ désigne l'ensemble des temps d'arrêt à valeur dans $[t,T]$. Nous obtenons le résultat suivant :

Théorème. *Les deux suites itératives v_n et φ_n coïncident et convergent vers la fonction de valeur initiale v :*

$$\begin{array}{rcl} \varphi_n(t,z) & = & v_n(t,z), \\ \lim_{n\to\infty} \varphi_n(t,z) & = & v(t,z), \quad (t,z) \in [0,T] \times \mathcal{S}. \end{array}$$

Etudes et résultats numériques. Compte tenu de la dimension de notre problème et surtout de la complexité de notre région de solvabilité, une résolution numérique par les méthodes des différences finies s'avère extrêmement fastidieuse. Nous choisissons ainsi les méthodes de Monte Carlo pour le calcul de la suite itérative

$$v_{n+1}(t,z) = \sup_{\tau \in \mathcal{S}_{t,T}} E\left[e^{-r(\tau-t)}\mathcal{H}v_n(\tau, X_\tau^{0,t,x}, y, P_\tau^{0,t,p})\right], \; z \in \bar{\mathcal{S}}$$

et les regions de transaction et de non-transaction. Elles consistent à discrétiser notre espace-temps et à calculer de nombreuses espérances conditionnelles associées qui représentent les approximations des fonctions de valeur v_n à chaque point de la grille. Pour cela, nous utilisons une méthode, basée sur le calcul de Malliavin, suggérée par Fournié, Lasry, Lebuchoux, Lions et Touzi [29] et développée par Bouchard, Ekeland et Touzi [10].

0.3 Options réelles et contrôle stochastique

0.3.1 Solution explicite à un problème de changement de régime optimal à deux états

Dans ce chapitre, on étudie la théorie d'arrêt optimal et sa généralisation appliquées au problème de changement de régime. Pour cela, on considère un processus stochastique de diffusion uni-dimensionnelle, X, qui peut prendre un nombre fini de régimes ou d'états. Les régimes peuvent être changés lors d'une suite de temps d'arrêt décidés par l'opérateur, avec des coûts fixes. Un exemple illustrant parfaitement cette étude est le problème d'investissement d'une firme dans un environnement incertain, où l'on gère plusieurs sites de production opérant dans différents modes ou régimes selon les différentes perspectives économiques. Le processus X représente le prix des matières premières consommées ou des biens produits et sa dynamique change selon le régime sous lequel il opère. Le projet de l'entreprise génère un flux selon une fonction de profit qui dépend du prix X et du choix de régime. Le problème est de trouver la stratégie optimale de changement de régime qui maximise l'espérance des profits résultant de ce projet.

Plusieurs auteurs ont traité le problème de changement de régime, Bensoussan et Lions [7] et Tang et Yong [64], ainsi que son application à l'évaluation d'options, aux options réelles et aux problèmes d'investissement dans un environnement incertain, Brekke et Oksendal [12], Duckworth et Zervos [27], Hamadène et Jeanblanc [37], et Guo [35]. Dans [37], les auteurs ont recours aux notions d'équations différentielles stochastiques rétrogrades et d'enveloppe de Snell pour résoudre un problème de changement de régime à deux états, correspondant aux états d'une centrale électrique : en fonctionnement ou à l'arrêt. Après avoir prouvé l'existence d'une stratégie optimale et en avoir fourni une expression, ils donnent également une méthode de simulation et quelques résultats numériques. Dans [12] et [27] traitant un problème à deux régimes, des solutions explicites ont été obtenues. Leur méthode de résolution est de construire une solution au système de la programmation dynamique en devinant la forme à priori de la stratégie optimale, puis de la valider à posteriori par vérification. Dans ces deux travaux, il n'y a pas de changement de régime dans le processus de diffusion car le changement de régime se résume au changement de fonction de profit.

Dans notre étude, nous considérons un modèle dont le changement de régime concerne à la fois le processus de diffusion et la fonction de profit.

Formulation du problème. On considère d'abord que le processus de diffusion X est un brownien géométrique et peut prendre un nombre fini de régimes. Chaque régime correspond à un couple de tendance et volatilité (b_i, σ_i). On modélise ce problème d'optimisation par une stratégie de contrôle impulsionnel $\alpha = (\tau_n, \kappa_n)_{n\in\mathbb{N}^*}$ où les τ_n représentent les temps d'intervention de l'opérateur et κ_n le nouveau régime à l'instant τ_n.

On pose g_{ij} comme coût (algébrique) de changement de régime i au régime j avec la convention $g_{ii} = 0$ et suppose que ces coûts satisfont les relations d'arbitrage suivantes :

$$g_{ik} \;<\; g_{ij} + g_{jk}, \quad \forall i \neq j,\; j \neq k \;\in \mathbb{I}_d. \tag{0.3.1}$$

Ces relations triangulaires signifient qu'il est toujours préférable, en terme de coût, de faire en une fois un seul investissement que de faire deux investissements successifs équivalents. Elles empêchent également tout arbitrage consistant à faire des aller-retour $i \leftrightarrow j$: $0 < g_{ij} + g_{ji}, \;\; \forall i \neq j$.

Le problème d'investissement. Quand l'état initial du système est (x, i), le profit espéré pour une stratégie de contrôle $\alpha = (\tau_n, \kappa_n)_{n\geq 1} \in \mathcal{A}$ donnée, est

$$J_i(x,\alpha) \quad = \quad E\left[\int_0^\infty e^{-rt} f(X_t^{x,i}, I_t^i)dt - \sum_{n=1}^\infty e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}\right].$$

Avec $r > 0$, le taux d'actualisation. Pour la suite de ce chapitre, on pose $f_i(.) = f(.,i)$.

L'objectif est de maximiser ce profit espéré sur toutes les strategies de $\mathcal{A}$. Ainsi, on définit les fonctions de valeur

$$v_i(x) \quad = \quad \sup_{\alpha\in\mathcal{A}} J_i(x,\alpha), \quad x \in \mathbb{R}_+^*, \ i \in \mathbb{I}_d. \tag{0.3.2}$$

On obtient la caractérisation par EDP des fonctions de valeur par les notions de solution de viscosité comme suit :

Théorème *Les fonctions de valeur v_i, $i \in \mathbb{I}_d$, sont les solutions uniques avec les conditions de croissance linéaire sur $(0,\infty)$ et les conditions au bord $v_i(0^+) = \max_{j\in\mathbb{I}_d}[-g_{ij}]$ au système d'inégalités variationnelles :*

$$\min\left\{rv_i - \mathcal{L}_i v_i - f_i \ , \ v_i - \max_{j\neq i}(v_j - g_{ij})\right\} \quad = \quad 0, \quad x \in (0,\infty), \quad i \in \mathbb{I}_d. \tag{0.3.3}$$

Résultats explicites pour un modèle à deux régimes. L'objectif principal est d'obtenir des solutions explicites dans le cas de modèle à deux régimes : l'expression des fonctions de valeur et la stratégie optimale correspondante.

Un recours aux notions de solution de viscosité s'avère être un outil puissant pour déterminer la solution au système d'inégalités variationnelles. On obtient ainsi directement la propriété de "smooth-fit" des fonctions de valeur et la structure des régions de "switching". On considère et obtient les solutions explicites dans les cas suivants :

- Le couple tendance et volatilité de la diffusion prend deux valeurs différentes selon les régimes, et les fonctions de profit sont identiques et de type puissance.
- Il n'y a pas de "switching" dans le processus de diffusion et les deux différentes fonctions de profit satisfont une condition générale, incluant les fonctions de type puissance.

Pour chacun des deux cas, on considère également les cas suivants : les deux coûts de "switching" sont positifs, et l'un des deux coûts est négatif. Ce dernier cas est, par ailleurs, très intéressant en terme d'applications où une firme choisit entre l'ouverture ou la fermeture d'une activité. Lors de la fermeture, la firme pourrait recouvrir une partie du coût de l'ouverture.

0.3.2 Un problème couplé de contrôle singulier et de changement de régime pour une politique de dividende avec investissement réversible

L'évaluation d'une entreprise est non seulement un problème fondamental en finance d'entreprise mais également un des piliers fondateurs du marché financier. Plusieurs méthodes sont utilisées par les intervenants des marchés d'actions, en particulier les analystes financiers, dont les plus fréquemment utilisées sont le "Discounted Cash Flow", les différents multiples tels que le "Price Earnings Ratio" et le "EBITDA multiple". Cependant, la valeur d'une entreprise provient théoriquement de sa capacité à générer des bénéfices afin de les distribuer aux actionnaires. Elle représente donc la valeur actualisée des dividendes futurs. La méthode de "Discounted Dividends Flow" est ainsi, parmi toutes les méthodes, la plus pertinente.

La valeur d'une entreprise dépend d'un ensemble de paramètres tels que le prix, la demande et le niveau de compétition, tous soumis aux aléas du marché dans lequel elle opère. Mais, elle dépend aussi et surtout de la capacité du manager à identifier et exécuter la meilleure politique de dividende et d'investissement maximisant l'intérêt des actionnaires. S'il ne peut fixer le niveau du "cash-flow" généré car soumis à un environnement incertain, il peut, par contre, fixer quasi-arbitrairement les niveaux de dividende et d'investissement, avec la faillite comme seule contrainte. Les meilleurs managers sont ceux qui arrivent à identifier cette politique optimale.

Depuis les années 90, des mathématiciens ont tenté de modéliser et de résoudre ces problèmes de gouvernance d'entreprise comme un problème d'optimisation et de contrôle stochastique. Parmi les premiers travaux sur la politique de dividende optimale, on peut mentionner ceux de Jeanblanc et Shiryaev [43] et de Choulli, Taksar et Zhou [18]. Mathématiquement, ces études sont formulées comme des problèmes de contrôle stochastique singulier. D'autres travaux sont portés sur la politique d'investissement optimale. Les théories sur la politique d'investissement, dans un environnement incertain pour une entreprise pouvant opérer les activités de production sous différents modes ou régimes, ont conduit aux recherches sur les problèmes de changement de régime ou "optimal switching problems", qui a récemment reçu beaucoup d'attention de la communauté des mathématiciens, voir Brekke et Oksendal [12], Duckworth et Zervos [27], Hamadène et Jeanblanc [37], Ly Vath et Pham [51].

Cependant, étudier séparément les deux points de recherche en finance d'entreprise, la politique optimale de dividende et d'investissement dans un environnement incertain, ne satisfait guère les réalités économiques d'une entreprise, compte tenu de la forte interaction entre ces deux facteurs. Notre étude porte donc sur le problème couplé de contrôle singulier et de changement de régime. Elle est une extension de l'étude faite par Décamps et Villeneuve [24], qui considèrent l'interaction entre la politique de dividende et d'investissement irréversible dans un environnement incertain. Notre but est de relaxer l'hypothèse d'irréversibilité de l'investissement, c'est-à-dire, de l'opportunité de croissance. Autrement dit, quand une entreprise, opérant sous une certaine technologie, a l'opportunité d'investir pour la croissance future dans une nouvelle technologie, elle peut décider, une fois cette technologie installée, de retourner dans l'ancienne technologie en recevant en compensation une partie du coût investi.

Notre étude est suffisamment riche pour adresser plusieurs questions posées dans la littérature des options réelles : les effets des contraintes financières sur les décisions d'investissement, quand est-il optimal de retarder la distribution de dividende afin d'investir...

Formulation du problème. La formulation mathématique de ce problème nous amène à considérer un problème couplé de contrôle singulier et de changement de régime pour une diffusion uni-dimensionnelle. Le processus de diffusion considéré, X, représente la dynamique de la réserve de cash :

$$dX_t \;=\; \mu_{I_t} dt + \sigma dW_t - dZ_t - dK_t, \quad X_{0^-} \;=\; x, \tag{0.3.4}$$

où μ_{I_t} représentent les quantités de cash générées par l'entreprise selon que l'on est sous le régime $I_t \in \{0,1\}$. Z représente les dividendes totaux distribués jusqu'à l'instant t alors que K est le coût lié aux décisions d'investissement et de désinvestissement.

On considère $g > 0$ le coût de l'investissement dans la nouvelle technologie : le passage du régime 0 au régime 1, tandis que le désinvestissement, du régime 1 au régime 0, apporte un cash de $(1-\lambda)g$, avec $0 < \lambda < 1$.

Le problème d'investissement. Notre objectif est de maximiser la valeur reçue par les actionnaires, c'est-à-dire, la somme actualisée des dividendes futurs reçus jusqu'à la perpétuité ou la faillite éventuelle de l'entreprise. On cherche donc à obtenir la valeur optimale de l'entreprise,

$$v_i(x) \;=\; \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right], \quad x \in \mathbb{R},\; i = 0,1, \tag{0.3.5}$$

où est T est l'instant de faillite de l'entreprise

$$T \;=\; T^{x,i,\alpha} \;=\; \inf\left\{t \geq 0 \;: X_t^{x,i} \;<\; 0\right\},$$

et éventuellement la politique de dividende et d'investissement optimale correspondante.

Résultats. Ce problème couplé nous amène via le principe de la programmation dynamique à un système d'inégalités variationnelles. On utilise pour cela l'approche de solution de viscosité. On obtient ainsi :

- la continuité des fonctions de valeur v_i, $i = 0,1$, et qu'elles sont l'unique solution de viscosité au système d'inégalités variationnelles associé.
- la régularité des fonctions de valeur : elles sont de classe C^1 sur $(0,\infty)$ et de C^2 sur l'union des régions de continuité et de distribution de dividende.

Le résultat majeur de notre étude est la caractérisation de l'intuition naturelle que le manager préfère retarder le paiement de dividende si l'investissement offre suffisamment d'opportunité de croissance. Nous obtenons qualitativement les régions de "switching" qui peuvent prendre différentes formes dépendant des taux de profit de chaque technologie et des coûts de transition. Les résultats ci-dessous donnent les descriptions qualitatives et explicites de la structure de la solution à notre problème de contrôle :

Résultats Principaux. *Nous distinguons les différents cas suivants :*

(i) *Si l'opportunité de croissance est trop faible, i.e.* $\mu_1 \leq Seuil_m$,

$\star$ *au régime* 0*, il est optimal de ne jamais investir,*

$\star$ *au régime* 1*, il est optimal de distribuer toute la réserve de cash comme dividende et de désinvestir et revenir au régime* 0.

(ii) *Si l'opportunité de croissance est quantitativement moyenne, i.e.* $Seuil_m < \mu_1 \leq Seuil_M$,

$\star$ *au régime* 0*, il est optimal de ne jamais investir,*

$\star$ *au régime* 1*, il est optimal de toujours rester dans ce régime quand l'entreprise n'est pas en faillite. Mais dès que l'on s'approche de la faillite, c'est-à-dire, quand* $x = 0$*, il faut désinvestir et revenir au régime* 0.

(iii) *Si l'opportunité est suffisamment forte, i.e.* $\mu_1 > Seuil_M$,

$\star$ *au régime* 1*, il est optimal de toujours rester dans ce régime quand l'entreprise n'est pas en faillite, par contre, lorsque l'on s'approche de la faillite, il faut désinvestir et revenir au régime* 0,

$\star$ *au régime* 0*, il faut distinguer deux cas :*

cas 1.) Il est optimal d'investir dès que le processus de réserve de cash dépasse un certain seuil x^*_{01}*, alors que dès qu'il passe sous un certain seuil* a*, il est optimal de distribuer comme dividende tout le cash excédant un certain seuil* $\hat{x}_0$ *et d'abandonner toute opportunité de croissance (avec* $\hat{x}_0 < a < x^*_{01}$*).*

cas 2.) Le manager retarde tout paiement de dividende afin d'investir dans la nouvelle technologie dès que la réserve de cash dépasse x^*_{01}.

0.4 Équilibre de marché compétitif sous asymétrie d'information

Les théories classiques des modèles du marché financier supposent que tous les intervenants du marché ont accès aux mêmes informations. Il est cependant clair que cette hypothèse ne correspond pas à la réalité du marché. Tous les intervenants n'ont pas accès aux mêmes informations, autrement dit, il y a une asymétrie d'information. L'asymétrie d'information peut s'avérer de plusieurs manières : certains ont accès aux informations confidentielles et non publiques, tandis que d'autres constituent, à partir d'un ensemble d'informations publiques et non-matérielles, des informations propriétaires et pertinentes pour les décisions d'investissement.

Ces dernières années, de nombreux mathématiciens s'intéressent aux problèmes posés par l'asymétrie d'information. En général, cette asymétrie d'information est modélisée par le fait que certains agents du marché possèdent des informations additionnelles à celles publiquement disponibles. Une information additionnelle pourrait être, par exemple, le futur prix de liquidation d'un actif risqué. Utilisant la théorie de grossissement de filtration développée par Jeulin [44] et puis Jacod [42], plusieurs études telles que celles de Pikovsky et Karatzas et de Grorud et Pontier [33] cherchent à résoudre des problèmes de maximisation dans un marché où deux investisseurs ont différents niveaux d'information. Les prix des actifs évoluent selon une diffusion exogène. Cependant, l'inconvenient des modèles ci-dessus est que l'agent ordinaire ne peut déduire du marché l'information additionnelle ou "insider information" que détient l'agent initié.

Par contre, dans Kyle [48] and Back [3], le marché est compétitif et l'agent ordinaire peut obtenir des "feedbacks" du marché concernant l'information additionnelle. Dans Biais et Rochet [8], où l'on peut trouver d'intéressantes études faites sur l'asymétrie d'information, l'objectif est d'analyser la formation de prix dans une version dynamique du modèle de Grossman et Stiglitz [34] et où les techniques de contrôle stochastique sont utilisées.

Dans le même cadre, notre étude considère un marché financier avec un actif risqué et un actif sans risque. Un agent ordinaire, un agent initié et des "noise-traders" forment l'ensemble des intervenants du marché. Si le premier ne peut observer que la dynamique du prix de l'actif risqué, S, le deuxième a, de plus, la connaissance de Z, l'offre totale de l'actif. Comme dans Back [3], en se basant sur l'observation du prix de l'actif risqué, l'agent ordinaire peut partiellement déduire l'information additionnelle de l'agent initié.

Tous deux possèdent une fonction d'utilité du type CARA. Chaque agent, consideré comme rationnel, cherche à maximiser l'espérance de l'utilité de sa richesse terminale.

Formulation du problème. On suppose que le processus Z est gouverné par l'e.d.s suivante :

$$dZ_t = (a(t)Z_t + b(t))\,dt + \gamma(t)dW_t, \quad Z_0 = z_0 \in \mathbb{R} \tag{0.4.1}$$

L'objectif de l'étude. Notre objectif est de déterminer si une condition d'équilibre peut être atteinte par un processus de prix linéaire, étant donné un processus linéaire Z. On définit comme admissible un processus de prix de la forme suivante :

$$dS_t \quad = \quad S_t\left[(\alpha(t)Z_t + \beta(t))\,dt + \sigma(t)dW_t\right], \qquad 0 \leq t \leq T \tag{0.4.2}$$

Résultats. Utilisant des techniques du contrôle stochastique et la théorie du filtrage, nous montrons que l'existence d'un équilibre de marché compétitif sous asymétrie d'information est directement liée à l'existence de solution d'un certain systéme d'équations non-linéaires. Cependant, on ne peut déterminer si l'ensemble des solutions de ce système d'équations est vide ou non.

Nous avons aussi entrepris l'étude d'un cas particulier où la dynamique de l'offre totale est un mouvement brownien. Nous avons montré que l'équilibre peut être atteint et obtenu explicitement la dynamique linéaire du processus de prix admissible correspondant.

Part I

STOCHASTIC CONTROL: A LIQUIDITY RISK MODEL

Chapter 1

A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact: Theoretical Aspect

Joint paper with Mohamed MNIF and Huyên PHAM, to appear in *Finance and Stochastics*

Abstract : We study a financial model with one risk-free and one risky asset subject to liquidity risk and price impact. In this market, an investor may transfer funds between the two assets at any discrete time. Each purchase or sale policy decision affects the price of the risky asset and incurs some fixed transaction cost. The objective is to maximize the expected utility from terminal liquidation value over a finite horizon and subject to a solvency constraint. This is formulated as an impulse control problem under state constraint and we characterize the value function as the unique constrained viscosity solution to the associated quasi-variational Hamilton-Jacobi-Bellman inequality.

Keywords: portfolio selection, liquidity risk, impulse control, state constraint, discontinuous viscosity solutions.

1.1 Introduction

Classical market models in mathematical finance assume perfect elasticity of traded assets : traders act as price takers, so that they buy and sell with arbitrary size without changing the price. However, the market microstructure literature has shown both theoretically and empirically that large trades move the price of the underlying assets. Moreover, in practice, investors face trading strategies constraints, typically of finite variation, and they cannot rebalance them continuously. We then usually speak about liquidity risk or illiquid markets. While the assumption of perfect liquidity market may not be practically important over a very long term horizon, price impact can have a significant difference over a short time horizon.

Several suggestions have been proposed to formalize the liquidity risk. In [48] and [3], the impact of trading strategies on prices is explained by the presence of an insider. In the market manipulation literature, prices are assumed to depend directly on the trading strategies. For instance, the paper [20] considers a diffusion model for the price dynamics with coefficients depending on the large investor's strategy, while [30], [58], [56], [4] or [14] develop a continuous-time model where prices depend on strategies via a reaction function. While the assumption of price-taker may not be practically important for investors making allocation decision over a very long time horizon, price impact can make a significant difference when investors execute large trades over a short time of horizon. The market microstructure literature has shown both theoretically and empirically that large trades move the price of the underlying securities. Moreover, it is also well established that transaction costs in asset markets are an important factor in determining the trading behavior of market participants; we mention among others [22] and [45] for the literature on arbitrage and optimal trading policies, and [65], [50] for the literature on the impact of transaction costs on agents' economic behavior. Consequently, transaction costs should affect market liquidity and asset prices. This is the point of view in the academic literature where liquidity is defined in terms of the bid-ask spread and/or transaction costs associated with a trading strategy. On the other hand, in the practitioner literature, illiquidity is often viewed as the risk that a trader may not be able to extricate himself from a position quickly when need arises. Such a situation occurs when continuous trading is not permitted, for instance, because of fixed transaction costs.

Of course, in actual markets, both aspects of market manipulation and transaction costs are correct and occur simultaneously. In this paper, we propose a model of liquidity risk and price impact that adopts both these perspectives. Our model is inspired from the recent papers [63] and [38], and may be described roughly as follows. Trading on illiquid assets is not allowed continuously due to some fixed costs but only at any discrete times. These liquidity constraints on strategies are in accordance with practitioner literature and consistent with the academic literature on fixed transaction costs, see e.g. [54]. There is an investor, who is large in the sense that his strategies affect asset prices : prices are pushed up when buying stock shares and moved down when selling shares. In this context, we study an optimal portfolio choice problem over a finite horizon : the investor maximizes his expected utility from terminal liquidation wealth and under a natural economic solvency constraint. In some sense, our problem may be viewed as a continuous-time version of the recent discrete-time one proposed in [15] . We mention also the paper [2], which studies an optimal trade execution problem in a discrete time setting with permanent and temporary market impact.

Our optimization problem is formulated as a parabolic impulse control problem with three variables (besides time variable) related to the cash holdings, number of stock shares and price. This problem is known to be associated by the dynamic programming principle to a Hamilton-Jacobi-Bellman (HJB) quasi-variational inequality, see [7]. We refer to [43], [47], [13] or [55] for some recent papers involving applications of impulse controls in finance, mostly over an infinite horizon and in dimension 1, except [47] and [55] in dimension 2. There is in addition, in our context, an important aspect related to the economic solvency condition requiring that liquidation wealth is nonnegative, which is translated into a state constraint involving a nonsmooth boundary domain. The model and the detailed description of the liquidation value and solvency region, and its formulation as an impulse control problem are exposed in Section 1.2. Our main goal is to obtain a rigorous characterization result on the value function through the associated HJB quasi-variational inequality. The main result is formulated in Section 1.3.

The features of our stochastic control problem make appear several technical difficulties related to the nonlinearity of the impulse transaction function and the solvency constraint. In particular, the liquidation net wealth may grow after transaction, which makes nontrivial the finiteness of the value function. Hence, the Merton bound does not provide as e.g. in transaction cost models, a natural upper bound on the value function. Instead, we provide a suitable "linearization" of the liquidation value that provides a sharp upper bound of the value function. The solvency region (or state domain) is not convex and its boundary even not smooth, in contrast with transaction cost model (see [22]), so that continuity of the value function is not direct. Moreover, the boundary of the solvency region is not absorbing as in transaction cost models and singular control problems, and the value function may be discontinuous on some parts of the boundary. Singularity of our impulse control problem appears also at the liquidation date, which translates into discontinuity of the value function at the terminal date. These properties of the value function are studied in Section 1.4.

In our general set-up, it is then natural to consider the HJB equation with the concept of (discontinuous) viscosity solutions, which provides by now a well established method for dealing with stochastic control problems, see e.g. the book [28]. More precisely, we need to consider constrained viscosity solutions to handle the state constraints. Our first main result is to prove that the value function is a constrained viscosity solution to its associated HJB quasi-variational inequality. Our second main result is a new comparison principle for the state constraint HJB quasi-variational inequality, which ensures a PDE characterization for the value function of our problem. Previous comparison results derived for variational inequality (see [40], [64]) associated to impulse problem do not apply here. In our context, we prove that one can compare a subsolution with a supersolution to the HJB quasi-variational inequality provided that one can compare them at the terminal date (as usual in parabolic problems) but also on some part D_0 of the solvency boundary, which represents an original point in comparison principle for state-constraint problem. Section 1.5 is devoted to the PDE viscosity characterization of the value function. We conclude in Section 1.6 with some remarks.

1.2 The Model

This section presents the details of the model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ supporting an one-dimensional Brownian motion W on a finite horizon $[0, T]$, $T < \infty$. We consider a continuous time financial market model consisting of a money market account yielding a constant interest rate $r \geq 0$ and a risky asset (or stock) of price process $P = (P_t)$. We denote by X_t the amount of money (or cash holdings) and by Y_t the number of shares in the stock held by the investor at time t.

Liquidity constraints. We assume that the investor can only trade discretely on $[0, T)$. This is modelled through an impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 1}$: $\tau_1 \leq \ldots \tau_n \leq \ldots < T$ are stopping times representing the intervention times of the investor and ζ_n, $n \geq 1$, are $\mathcal{F}_{\tau_n}$-measurable random variables valued in $\mathbb{R}$ and giving the number of stock purchased if $\zeta_n \geq 0$ or sold if $\zeta_n < 0$ at these times. The sequence (τ_n, ζ_n) may be a priori finite or infinite. The dynamics of Y is then given by :

$$
\begin{aligned}
Y_s &= Y_{\tau_n}, \qquad \tau_n \leq s < \tau_{n+1} && (1.2.1)\\
Y_{\tau_{n+1}} &= Y_{\tau_n} + \zeta_{n+1}. && (1.2.2)
\end{aligned}
$$

Notice that we do not allow trade at the terminal date T, which is the liquidation date.

Price impact. The large investor affects the price of the risky stock P by his purchases and sales : the stock price goes up when the trader buys and goes down when he sells and the impact is increasing with the size of the order. We then introduce a price impact positive function $Q(\zeta, p)$ which indicates the post-trade price when the large investor trades a position of ζ shares of stock at a pre-trade price p. In absence of price impact, we have $Q(\zeta, p) = p$. Here, we have $Q(0, p) = p$ meaning that no trading incurs no impact and Q is nondecreasing in ζ with $Q(\zeta, p) \geq$ (resp. $\leq$) p for $\zeta \geq$ (resp. $\leq$) 0. Actually, in the rest of the paper, we consider a price impact function in the form

$$
Q(\zeta, p) = pe^{\lambda \zeta}, \qquad \text{where } \lambda > 0. \tag{1.2.3}
$$

The proportionality factor $e^{\lambda \zeta}$ represents the price increase (resp. discount) due to the ζ shares bought (resp. sold). The positive constant λ measures the fact that larger trades generate larger quantity impact, everything else constant. This form of price impact function is consistent with both the asymmetric information and inventory motives in the market microstructure literature (see [48]).

We then model the dynamics of the price impact as follows. In the absence of trading, the price process is governed by

$$dP_s = P_s(bds + \sigma dW_s), \qquad \tau_n \leq s < \tau_{n+1}, \tag{1.2.4}$$

where b, σ are constants with $\sigma > 0$. When a discrete trading $\Delta Y_s := Y_s - Y_{s^-} = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$, the price jumps to $P_s = Q(\Delta Y_s, P_{s^-})$, i.e.

$$P_{\tau_{n+1}} = Q(\zeta_{n+1}, P_{\tau_{n+1}^-}). \tag{1.2.5}$$

Notice that with this modelling of price impact, the price process P is always strictly positive, i.e. valued in $\mathbb{R}_+^* = (0, \infty)$.

Cash holdings. We denote by $\theta(\zeta, p)$ the cost function, which indicates the amount for a (large) investor to buy or sell ζ shares of stock when the pre-trade price is p :

$$\theta(\zeta, p) = \zeta Q(\zeta, p).$$

In absence of transactions, the process X grows deterministically at exponential rate r :

$$dX_s = rX_s ds, \qquad \tau_n \leq s < \tau_{n+1}. \tag{1.2.6}$$

When a discrete trading $\Delta Y_s = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$ with pre-trade price $P_{s^-} = P_{\tau_{n+1}^-}$, we assume that in addition to the amount of stocks $\theta(\Delta Y_s, P_{s^-}) = \theta(\zeta_{n+1}, P_{\tau_{n+1}^-})$, there is a fixed cost $k > 0$ to be paid. This results in a variation of cash holdings by $\Delta X_s := X_s - X_{s^-} = -\theta(\Delta Y_s, P_{s^-}) - k$, i.e.

$$X_{\tau_{n+1}} = X_{\tau_{n+1}^-} - \theta(\zeta_{n+1}, P_{\tau_{n+1}^-}) - k. \tag{1.2.7}$$

The assumption that any trading incurs a fixed cost of money to be paid will rule out continuous trading, i.e. optimally, the sequence (τ_n, ζ_n) is not degenerate in the sense that for all n, $\tau_n < \tau_{n+1}$ and $\zeta_n \neq 0$ a.s. A similar modelling of fixed transaction costs is considered in [54] and [47].

Liquidation value and solvency constraint. The solvency constraint is a key issue in portfolio/consumption choice problem. The point is to define in an economically meaningful way what is the portfolio value of a position in cash and stocks. In our context, we introduce the liquidation function $\ell(y, p)$ representing the value that an investor would obtained by liquidating immediately his stock position y by a single block trade, when the pre-trade price is p. It is given by :

$$\ell(y, p) = -\theta(-y, p).$$

If the agent has the amount x in the bank account, the number of shares y of stocks at the pre-trade price p, i.e. a state value $z = (x, y, p)$, his net wealth or liquidation value is given by :

$$L(z) \quad = \quad \max[L_0(z), L_1(z)]1_{y\geq 0} + L_0(z)1_{y<0}, \tag{1.2.8}$$

where

$$L_0(z) \ = \ x + \ell(y,p) - k, \qquad L_1(z) \ = \ x.$$

The interpretation is the following. $L_0(z)$ corresponds to the net wealth of the agent when he liquidates his position in stock. Moreover, if he has a long position in stock, i.e. $y \geq 0$, he can also choose to bin his stock shares, by keeping only his cash amount, which leads to a net wealth $L_1(z)$. This last possibility may be advantageous, i.e. $L_1(z) \geq L_0(z)$, due to the fixed cost k. Hence, globally, his net wealth is given by (1.2.8). In the absence of liquidity risk, i.e $\lambda = 0$, and fixed transaction cost, i.e. $k = 0$, we recover the usual definition of wealth $L(z) = x + py$. Our definition (1.2.8) of liquidation value is also consistent with the one in transaction costs models where portfolio value is measured after stock position is liquidated and rebalanced in cash, see e.g. [21] and [55]. Another alternative would be to measure the portfolio value separately in cash and stock as in [25] for transaction costs models. This study would lead to multidimensional utility functions and is left for future research.

We then naturally introduce the liquidation solvency region (see Figure 1) :

$$\mathcal{S} \quad = \quad \left\{ z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : L(z) > 0 \right\},$$

and we denote its boundary and its closure by

$$\partial\mathcal{S} \ = \ \left\{ z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : L(z) = 0 \right\} \quad \text{and} \quad \bar{\mathcal{S}} \ = \ \mathcal{S} \cup \partial\mathcal{S}.$$

Remark 1.2.1 The function L is clearly continuous on $\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : y \neq 0\}$. It is discontinuous on $z_0 = (x, 0, p) \in \bar{\mathcal{S}}$, but it is easy to check that it is upper-semicontinuous on z_0, so that globally L is upper-semicontinuous. Hence $\bar{\mathcal{S}}$ is closed in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^*$. We also notice that L is nonlinear in the state variables, which contrasts with transaction costs models.

Remark 1.2.2 For any $p > 0$, the function $y \mapsto \ell(y,p) = pye^{-\lambda y}$ is increasing on $[0, 1/\lambda]$, decreasing on $[1/\lambda, \infty)$ with $l(0,p) = \lim_{y\to\infty} l(y,p) = 0$ and $l(1/\lambda, p) = pe^{-1}/\lambda$. We then distinguish the two cases :

$\star$ if $p < k\lambda e$, then $l(y,p) < k$ for all $y \geq 0$.

$\star$ if $p \geq k\lambda e$, then there exists an unique $y_1(p) \in (0, 1/\lambda]$ and $y_2(p) \in [1/\lambda, \infty)$ such that $l(y_1(p),p) = l(y_2(p),p) = k$ with $l(y,p) < k$ for all $y \in [0, y_1(p)) \cup (y_2(p), \infty)$. Moreover,

$y_1(p)$ (resp. $y_2(p)$) decreases to 0 (resp. increases to ∞) when p goes to infinity, while $y_1(p)$ (resp.$y_2(p)$) increases (resp. decreases) to $1/\lambda$ when p decreases to $k\lambda e$.
The boundary of the solvency region may then be explicitly obtained as follows (see Figures 2 and 3) :

$$\partial\mathcal{S} = \partial_\ell^-\mathcal{S} \cup \partial^y\mathcal{S} \cup \partial_0^x\mathcal{S} \cup \partial_1^x\mathcal{S} \cup \partial_2^x\mathcal{S} \cup \partial_\ell^+\mathcal{S},$$

where

$$\begin{aligned}
\partial_\ell^-\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : x + \ell(y,p) = k,\ y \leq 0\big\} \\
\partial^y\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : 0 \leq x < k,\ y = 0\big\} \\
\partial_0^x\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : x = 0,\ y > 0,\ p < k\lambda e\big\} \\
\partial_1^x\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : x = 0,\ 0 < y < y_1(p)),\ p \geq k\lambda e\big\} \\
\partial_2^x\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : x = 0,\ y > y_2(p),\ p \geq k\lambda e\big\} \\
\partial_\ell^+\mathcal{S} &= \big\{z = (x,y,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^* \ : x + \ell(y,p) = k,\ y_1(p) \leq y \leq y_2(p),\ p \geq k\lambda e\big\}.
\end{aligned}$$

In the sequel, we also introduce the corner lines in $\partial\mathcal{S}$:

$$\begin{aligned}
D_0 &= \{(0,0)\}\times\mathbb{R}_+^* \ \subset\ \partial^y\mathcal{S}, \qquad D_k = \{(k,0)\}\times\mathbb{R}_+^* \ \subset\ \partial_\ell^-\mathcal{S} \\
C_1 &= \{(0,y_1(p),p)\ : p\in\mathbb{R}_+^*\} \ \subset\ \partial_\ell^+\mathcal{S}, \qquad C_2 = \{(0,y_2(p),p)\ : p\in\mathbb{R}_+^*\} \ \subset\ \partial_\ell^+\mathcal{S}.
\end{aligned}$$

Admissible controls. Given $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$ and an initial state $Z_{t^-} = z$, we say that the impulse control strategy $\alpha = (\tau_n,\zeta_n)_{n\geq 1}$ is admissible if the process $Z_s = (X_s,Y_s,P_s)$ given by (1.2.1)-(1.2.2)-(1.2.4)-(1.2.5)-(1.2.6)-(1.2.7) (with the convention $\tau_0 = t$) lies in $\bar{\mathcal{S}}$ for all $s \in [t,T]$. We denote by $\mathcal{A}(t,z)$ the set of all such policies. We shall see later that this set of admissible controls is nonempty for all $(t,z) \in [0,T]\times\bar{\mathcal{S}}$.

Remark 1.2.3 We recall that we do not allow intervention time at T, which is the liquidation date. This means that for all $\alpha \in \mathcal{A}(t,z)$, the associated state process Z is continuous at T, i.e. $Z_{T^-} = Z_T$.

In the sequel, for $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$, we also denote $Z_s^{0,t,z} = (X_s^{0,t,x}, y, P_s^{0,t,p})$, $t \leq s \leq T$, the state process when no transaction (i.e. no impulse control) is applied between t and T, i.e. the solution to :

$$dZ_s^0 = \begin{pmatrix} rX_s^0 \\ 0 \\ bP_s^0 \end{pmatrix} ds + \begin{pmatrix} 0 \\ 0 \\ \sigma P_s^0 \end{pmatrix} dW_s, \tag{1.2.9}$$

starting from z at time t.

Investment problem. We consider an utility function U from $\mathbb{R}_+$ into $\mathbb{R}$, strictly increasing, concave and w.l.o.g. $U(0) = 0$, and s.t. there exist $K \geq 0$, $\gamma \in [0,1)$:

$$U(w) \leq Kw^\gamma, \quad \forall w \geq 0, \tag{1.2.10}$$

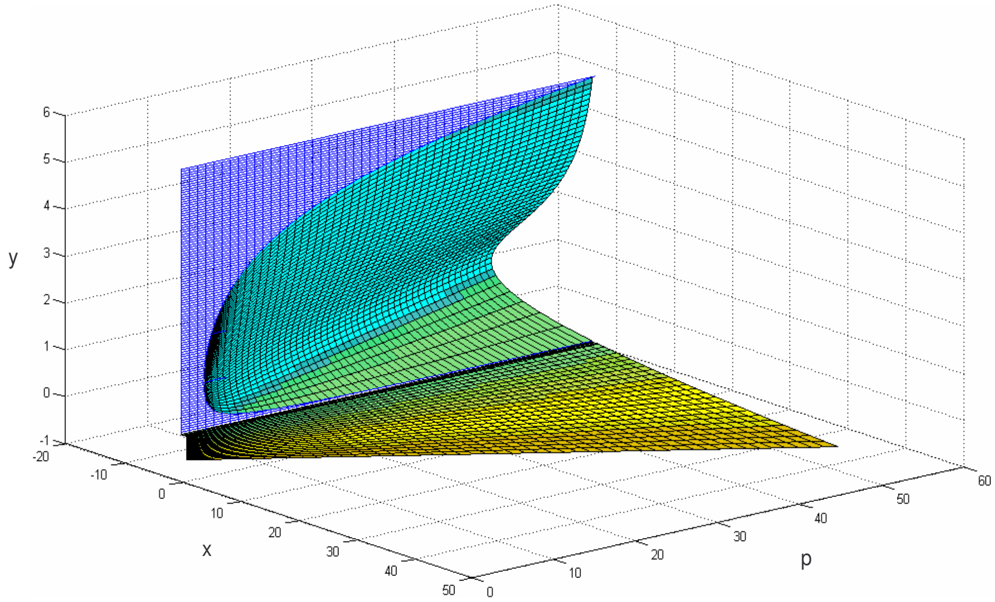


Figure 1: The solvency region when k = 1, λ = 1

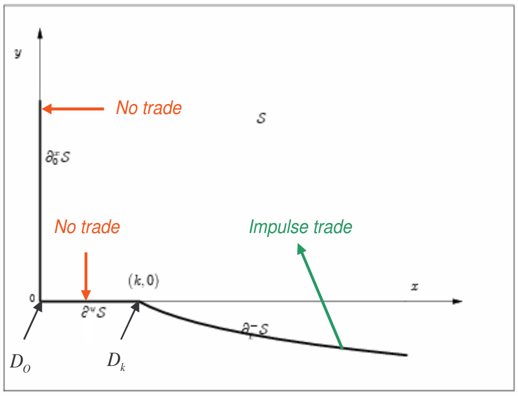


Figure 2: The solvency region when p < kλe

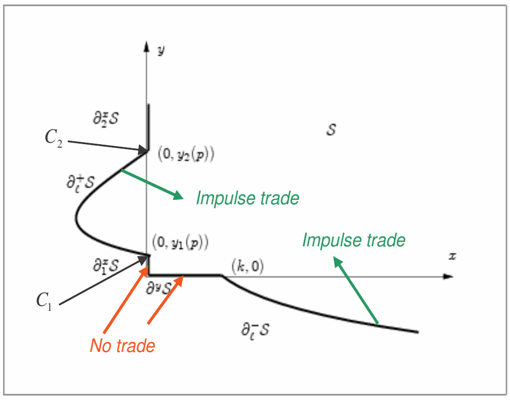


Figure 3: The solvency region when p > kλe

We denote U_L the function defined on $\bar{\mathcal{S}}$ by :

$$U_L(z) \quad = \quad U(L(z)).$$

We study the problem of maximizing the expected utility from terminal liquidation wealth and we then consider the value function :

$$v(t,z) \quad = \quad \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[U_L(Z_T)\right], \qquad (t,z) \in [0,T] \times \bar{\mathcal{S}}. \tag{1.2.11}$$

Remark 1.2.4 We shall see later that for all $\alpha \in \mathcal{A}(t,z) \neq \emptyset$, $U_L(Z_T)$ is integrable so that the expectation in (1.2.11) is finite. Since U is nonnegative and nondecreasing, we immediately get a lower bound for the value function :

$$v(t,z) \quad \geq \quad U(0) = 0, \qquad \forall t \in [0,T],\ z = (x,y,p) \in \bar{\mathcal{S}}.$$

We shall also see later that the value function v is finite in $[0,T] \times \bar{\mathcal{S}}$ by providing a sharp upper bound.

Notice that in contrast to financial models without frictions or with proportional transaction costs, the dynamics of the state process $Z = (X,Y,P)$ is nonlinear and then the value function v does not inherit the concavity property of the utility function. The solvency region is even not convex. In particular, one cannot derive as usual the continuity of the value function as a consequence of the concavity property. Moreover, for power-utility functions $U(w) = Kw^{\gamma}$, the value function does not inherit the homogeneity property of the utility function.

We shall adopt a dynamic programming approach to study this utility maximization problem. We end this section by recalling the dynamic programming principle for our stochastic control problem.

Dynamic programming principle (DPP). For all $(t,z) \in [0,T) \times \bar{\mathcal{S}}$, we have

$$v(t,z) \quad = \quad \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[v(\tau, Z_\tau)\right], \tag{1.2.12}$$

where $\tau = \tau(\alpha)$ is any stopping time valued in $[t,T]$ depending on α in (1.2.12). The precise meaning is :

(i) for all $\alpha \in \mathcal{A}(t,z)$, for all $\tau \in \mathcal{T}_{t,T}$, set of stopping times valued in $[t,T]$:

$$\mathbb{E}[v(\tau, Z_\tau)] \quad \leq \quad v(t,z) \tag{1.2.13}$$

(ii) for all $\varepsilon > 0$, there exists $\hat{\alpha}^\varepsilon \in \mathcal{A}(t,z)$ s.t. for all $\tau \in \mathcal{T}_{t,T}$:

$$v(t,z) \quad \leq \quad \mathbb{E}[v(\tau, \hat{Z}^\varepsilon_\tau)] + \varepsilon. \tag{1.2.14}$$

Here $\hat{Z}^\varepsilon$ denotes the state process starting from z at t and controlled by $\hat{\alpha}^\varepsilon$.

1.3 Quasi-variational Hamilton-Jacobi-Bellman inequality and main result

In this section, we introduce some notations, recall the dynamic programming quasi-variational inequality associated to the impulse control problem (1.2.11) and formulate the main result.

We define the impulse transaction function from $\bar{\mathcal{S}} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$:

$$\Gamma(z,\zeta) \quad = \quad (x - \theta(\zeta,p) - k, y + \zeta, Q(\zeta,p)), \quad z = (x,y,p) \in \bar{\mathcal{S}},\ \zeta \in \mathbb{R},$$

This corresponds to an immediate trading at time t of ζ shares of stock, so that from (1.2.2)-(1.2.5)-(1.2.7) the state process jumps from $Z_{t^-} = z \in \bar{\mathcal{S}}$ to $Z_t = \Gamma(z,\zeta)$. We then consider the set of admissible transactions :

$$\mathcal{C}(z) \quad = \quad \left\{\zeta \in \mathbb{R}\ : \Gamma(z,\zeta)\ \in\ \bar{\mathcal{S}}\right\} \ = \ \{\zeta \in \mathbb{R}\ : L(\Gamma(z,\zeta))\ \geq\ 0\},$$

in accordance with the solvency constraint and the set of admissibles controls $\mathcal{A}(t,z)$. We introduce the impulse operator $\mathcal{H}$ defined by :

$$\mathcal{H}\varphi(t,z) \quad = \quad \sup_{\zeta \in \mathcal{C}(z)} \varphi(t,\Gamma(z,\zeta)), \quad (t,z) \in [0,T] \times \bar{\mathcal{S}},$$

for any measurable function φ on $[0,T] \times \bar{\mathcal{S}}$. If for some $z \in \bar{\mathcal{S}}$, the set $\mathcal{C}(z)$ is empty, we denote by convention $\mathcal{H}\varphi(t,z) = -\infty$.

We also define $\mathcal{L}$ as the infinitesimal generator associated to the system (1.2.9) corresponding to a no-trading period :

$$\mathcal{L}\varphi \quad = \quad rx\frac{\partial \varphi}{\partial x} + bp\frac{\partial \varphi}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 \varphi}{\partial p^2}.$$

The HJB quasi-variational inequality arising from the dynamic programming principle (1.2.12) is then written as :

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v\ ,\ v - \mathcal{H}v\right] \quad = \quad 0, \quad \text{on} \quad [0,T) \times \mathcal{S}. \tag{1.3.1}$$

This divides the time-space liquidation solvency region $[0,T) \times \mathcal{S}$ into a *no-trade region*

$$\mathbf{NT} \quad = \quad \{(t,z) \in [0,T) \times \mathcal{S}\ :\ v(t,z)\ >\ \mathcal{H}v(t,z)\},$$

and a *trade region*

$$\mathbf{T} \quad = \quad \{(t,z) \in [0,T) \times \mathcal{S}\ : v(t,z)\ =\ \mathcal{H}v(t,z)\}.$$

The rigorous characterization of the value function through the quasi-variational inequality (1.3.1) together with the boundary and terminal conditions is stated by means of constrained viscosity solutions. Our main result is the following theorem, which follows from the results proved in Sections 1.4 and 1.5.

Theorem 1.3.1 *The value function v is continuous on $[0,T) \times \mathcal{S}$ and is the unique (in $[0,T) \times \mathcal{S}$) constrained viscosity solution to* (1.3.1) *satisfying the boundary and terminal condition :*

$$\lim_{\substack{(t',z') \to (t,z) \\ z' \in \mathcal{S}}} v(t',z') = 0, \quad \forall (t,z) \in [0,T) \times D_0 \tag{1.3.2}$$

$$\lim_{\substack{(t,z') \to (T,z) \\ t < T,\, z' \in \mathcal{S}}} v(t,z') = \max[U_L(z), \mathcal{H}U_L(z)], \quad \forall z \in \bar{\mathcal{S}}, \tag{1.3.3}$$

and the growth condition :

$$|v(t,z)| \le K\left(1 + \left(x + \frac{p}{\lambda}\right)\right)^{\gamma}, \quad \forall (t,z) \in [0,T) \times \mathcal{S} \tag{1.3.4}$$

for some positive constant $K < \infty$.

Remark 1.3.1 Continuity and uniqueness of the value function for the HJBQVI (1.3.1) hold true in $[0,T) \times \mathcal{S}$ in the class of functions satisfying the growth condition (1.3.4), associated to the terminal condition (1.3.3) (as usual in parabolic problems) but also to some specific boundary condition (1.3.2). This last point is nonstandard in constrained control problems, where one gets usually an uniqueness result for constrained viscosity solutions to the corresponding Bellman equation without any additional boundary condition, see e.g. [67] or [55]. Here, we have to impose a boundary condition on the non-smooth part D_0 of the solvency boundary. Notice also that the terminal condition is not given by U_L. Actually, it takes into account the fact that just before the liquidation date T, one can do an impulse transaction : the effect is to lift-up the utility function U_L through the impulse transaction operator $\mathcal{H}$.

1.4 Properties of the value function

1.4.1 Some properties on the impulse transactions set

In order to show that the value function of problem (1.2.11) is finite, which is not trivial a priori, we need to derive some preliminary properties on the set of admissible transactions $\mathcal{C}(z)$. Starting from a current state $z = (x,y,p) \in \bar{\mathcal{S}}$, an immediate transaction of size ζ leads to a new state $z' = (x',y',p') = \Gamma(z,\zeta)$. Recalling the expression (1.2.3) of the price impact function, we then have :

$$\begin{aligned} L_0(\Gamma(z,\zeta)) \;=\; x' + \ell(y',p') - k &= x + \ell(y,p) - k + p\zeta(e^{-\lambda y} - e^{\lambda\zeta}) - k \\ &= L_0(z) + pg(y,\zeta) - k, \end{aligned} \tag{1.4.1}$$

with

$$g(y,\zeta) = \zeta(e^{-\lambda y} - e^{\lambda\zeta}). \tag{1.4.2}$$

It then appears that due to the nonlinearity of the price impact function, and in contrast with transaction costs models, the net wealth may grow after some transaction : $L(\Gamma(z,\zeta))$ $> L(z)$ for some $z \in \bar{\mathcal{S}}$ and $\zeta \in \mathcal{C}(z)$. We first state the following useful result.

Lemma 1.4.1 *For all $z \in \bar{\mathcal{S}}$, the set $\mathcal{C}(z)$ is compact, eventually empty. We have :*

$$
\begin{aligned}
\mathcal{C}(z) &= \emptyset \quad \text{if } z \in \partial^y\mathcal{S} \cup \partial_0^x\mathcal{S} \cup \partial_1^x\mathcal{S}, \\
-\frac{1}{\lambda} \in \mathcal{C}(z) &\subset (-y, 0) \quad \text{if } z \in \partial_2^x\mathcal{S}, \\
-y \in \mathcal{C}(z) &\subset \begin{cases} [0,-y] & \text{if } z \in \partial_\ell^-\mathcal{S} \\ [-y,0) & \text{if } z \in \partial_\ell^+\mathcal{S} \end{cases}
\end{aligned}
$$

Moreover,

$$
\mathcal{C}(z) = \{-y\} \quad \text{if } z \in (\partial_\ell^-\mathcal{S} \cup \partial_\ell^{+,\lambda}\mathcal{S}) \cap \mathcal{N}_\ell
$$

where

$$
\partial_\ell^{+,\lambda}\mathcal{S} = \partial_\ell^+\mathcal{S} \cap \left\{ z \in \bar{\mathcal{S}} \ : y \le \frac{1}{\lambda} \right\}, \qquad \mathcal{N}_\ell = \left\{ z \in \bar{\mathcal{S}} \ : p\bar{g}(y) < k \right\},
$$

and $\bar{g}(y) = \max_{\zeta\in\mathbb{R}} g(y,\zeta)$.

The proof is based on detailed and long but elementary calculations on the liquidation net wealth $L(\Gamma(z,\zeta)) = \max\left[L_0(\Gamma(z,\zeta)), L_1(\Gamma(z,\zeta))\right] 1_{y+\zeta\ge 0} + L_0(\Gamma(z,\zeta))1_{y+\zeta<0}$ and is rejected in Appendix.

Remark 1.4.1 Actually, we have a more precise result on the compactness result of $\mathcal{C}(z)$. Let $z \in \bar{\mathcal{S}}$ and $(z_n)_n$ be a sequence in $\bar{\mathcal{S}}$ converging to z. Consider any sequence $(\zeta_n)_n$ with $\zeta_n \in \mathcal{C}(z_n)$, i.e. $L(\Gamma(z_n,\zeta_n)) \ge 0$:

$$
\begin{aligned}
&\max\left[L_0(z_n) + p_n g(y_n,\zeta_n) - k, x - \theta(\zeta_n,p_n) - k\right] 1_{y_n+\zeta_n\ge 0} \\
&\qquad\qquad + \left[L_0(z_n) + p_n g(y_n,\zeta_n) - k\right] 1_{y_n+\zeta_n<0} \ \ge \ 0.
\end{aligned}
$$

Since $g(y,\zeta)$ and $-\theta(\zeta,p)$ goes to $-\infty$ as ζ goes to infinity, and $g(y,\zeta)$ goes to $-\infty$ as ζ goes to $-\infty$, this proves that the sequence (ζ_n) is bounded. Hence, up to a subsequence, (ζ_n) converges to some $\zeta \in \mathbb{R}$. Since the function L is upper-semicontinuous, we see that the limit ζ satisfies : $L(\Gamma(z,\zeta)) \ge 0$, i.e. ζ lies in $\mathcal{C}(z)$.

We can now check that the set of admissible controls is not empty.

Corollary 1.4.1 *For all $(t,z) \in [0,T) \times \bar{\mathcal{S}}$, we have $\mathcal{A}(t,z) \neq \emptyset$.*

Proof. By continuity of the process $Z_s^{0,t,z}$, $t \le s \le T$, it is clear that it suffices to prove $\mathcal{A}(t,z) \neq \emptyset$ for any $t \in [0,T) \times \partial\mathcal{S}$. Fix now some arbitrary $t \in [0,T)$. From Lemma 1.4.1, the set of admissible transactions $\mathcal{C}(z)$ contains at least one nonzero element for any $z \in$

$\partial_2^x \mathcal{S} \cup \partial_\ell^+ \mathcal{S} \cup \partial_\ell^- \mathcal{S} \setminus D_k$. So once the state process reaches this boundary part, it is possible to jump inside the open solvency region $\mathcal{S}$. Hence, we only have to check that $\mathcal{A}(t,z)$ is nonempty when $z \in \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S} \cup \partial^y \mathcal{S} \cup D_k$. This is clear when $z \in \partial^y \mathcal{S} \cup D_k$: indeed, by doing nothing the state process $Z_s = Z_s^{0,t,z} = (xe^{r(s-t)}, 0, P_s^{0,t,p})$, $t \leq s \leq T$, obviously stays in $\bar{\mathcal{S}}$, since $x \geq 0$ and so $L_1(Z_s) \geq 0$ for all $t \leq s \leq T$. Similarly, when $z \in \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S}$, by doing nothing the state process $Z_s = Z_s^{0,t,z} = (0, y, P_s^{0,t,p})$, $t \leq s \leq T$, also stays in $\bar{\mathcal{S}}$ since $y \geq 0$ and so $L_1(Z_s) \geq 0$ for all $t \leq s \leq T$. □

We next turn to the finiteness of the value function, which is not trivial due to the impulse control. As mentioned above, since the net wealth may grow after transaction due to the nonlinearity of the liquidation function, one cannot bound the value function v by the value function of the Merton problem with liquidated net wealth. We then introduce a suitable "linearization" of the net wealth by defining the following functions on $\bar{\mathcal{S}}$:

$$\tilde{L}(z) \; = \; x + \frac{p}{\lambda}(1 - e^{-\lambda y}), \quad \text{and} \quad \bar{L}(z) \; = \; x + \frac{p}{\lambda}, \quad z = (x, y, p) \in \bar{\mathcal{S}}.$$

Lemma 1.4.2 *For all $z = (x, y, p) \in \bar{\mathcal{S}}$, we have :*

$$0 \; \leq \; L(z) \quad \leq \quad \tilde{L}(z) \; \leq \; \bar{L}(z) \tag{1.4.3}$$

and for all $\zeta \in \mathcal{C}(z)$

$$\tilde{L}(\Gamma(z,\zeta)) \quad \leq \quad \tilde{L}(z) - k \tag{1.4.4}$$
$$\bar{L}(\Gamma(z,\zeta)) \quad \leq \quad \bar{L}(z) - k. \tag{1.4.5}$$

In particular, we have $\mathcal{C}(z) = \emptyset$ for all $z \in \tilde{\mathcal{N}} := \{z \in \mathcal{S} \; : \tilde{L}(z) < k\}$.

Proof. 1) The inequality $\tilde{L} \leq \bar{L}$ is clear. Notice that for all $y \in \mathbb{R}$, we have

$$0 \quad \leq \quad 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}. \tag{1.4.6}$$

This immediately implies for all $z = (x, y, p) \in \bar{\mathcal{S}}$,

$$L_0(z) \quad \leq \quad \tilde{L}(z). \tag{1.4.7}$$

If $y \geq 0$, we obviously have $L_1(z) = x \leq \tilde{L}(z)$ and so $L(z) \leq \tilde{L}(z)$. If $y < 0$, then $L(z) = L_0(z) \leq \tilde{L}(z)$ by (1.4.7).

2) For any $z = (x, y, p) \in \bar{\mathcal{S}}$ and $\zeta \in \mathbb{R}$, a straightforward computation shows that

$$\tilde{L}(\Gamma(z,\zeta)) \quad = \quad \tilde{L}(z) - k + \frac{p}{\lambda}(e^{\lambda\zeta} - 1 - \lambda\zeta e^{\lambda\zeta}) \; \leq \; \tilde{L}(z) - k,$$

from (1.4.6). Similarly, we show (1.4.5). Finally, if $z \in \tilde{\mathcal{N}}$, we have from (1.4.5), $\tilde{L}(\Gamma(z,\zeta))$ < 0 for all $\zeta \in \mathcal{C}(z)$, which shows with (1.4.3) that $\mathcal{C}(z) = \emptyset$. □

As a first direct corollary, we check that the no-trade region is not empty.

Corollary 1.4.2 *We have* $\mathbf{NT} \neq \emptyset$. *More precisely, for each* $t \in [0, T)$, *the* t*-section of* $\mathbf{NT}$, *i.e.* $\mathbf{NT}(t) = \{z \in \mathcal{S} \ : (t, z) \in \mathbf{NT}\}$ *contains the nonempty subset* $\tilde{\mathcal{N}}$ *of* $\mathcal{S}$.

Proof. This follows from the fact that for any z lying in the nonempty set $\tilde{\mathcal{N}}$ of $\mathcal{S}$, we have $\mathcal{C}(z) = \emptyset$. In particular, $\mathcal{H}v(t, z) = -\infty < v(t, z)$ for $(t, z) \in [0, T) \times \tilde{\mathcal{N}}$. □

As a second corollary, we have the following uniform bound on the controlled state process.

Corollary 1.4.3 *For any* $(t, z) \in [0, T] \times \bar{\mathcal{S}}$, *we have almost surely for all* $t \leq s \leq T$:

$$\sup_{\alpha \in \mathcal{A}(t,z)} L(Z_s) \leq \sup_{\alpha \in \mathcal{A}(t,z)} \tilde{L}(Z_s) \leq \tilde{L}(Z_s^{0,t,z}) = X_s^{0,t,x} + \frac{P_s^{0,t,p}}{\lambda}(1 - e^{-\lambda y}), \quad (1.4.8)$$

$$\sup_{\alpha \in \mathcal{A}(t,z)} L(Z_s) \leq \sup_{\alpha \in \mathcal{A}(t,z)} \bar{L}(Z_s) \leq \bar{L}(Z_s^{0,t,z}) = X_s^{0,t,x} + \frac{P_s^{0,t,p}}{\lambda}, \quad (1.4.9)$$

$$\sup_{\alpha \in \mathcal{A}(t,z)} |X_s| \leq \frac{e}{e-1}\bar{L}(Z_s^{0,t,z}), \quad (1.4.10)$$

$$\sup_{\alpha \in \mathcal{A}(t,z)} P_s \leq \frac{\lambda e}{e-1}\bar{L}(Z_s^{0,t,z}). \quad (1.4.11)$$

Proof. Fix $(t, z) \in [0, T] \times \bar{\mathcal{S}}$ and consider for any $\alpha \in \mathcal{A}(t, z)$, the process $\tilde{L}(Z_s)$, $t \leq s \leq T$, which is nonnegative by (1.4.3). When a transaction occurs at time s, we deduce from (1.4.4) that the variation $\Delta\tilde{L}(Z_s) = \tilde{L}(Z_s) - \tilde{L}(Z_{s^-})$ is always negative : $\Delta\tilde{L}(Z_s) \leq -k \leq 0$. Therefore, the process $\tilde{L}(Z_s)$ is smaller than its continuous part :

$$L(Z_s) \leq \tilde{L}(Z_s) \leq \tilde{L}(Z_s^{0,t,z}), \quad t \leq s \leq T, \ a.s. \quad (1.4.12)$$

which proves (1.4.8) from the arbitrariness of α. Relation (1.4.9) is proved similarly.

From the second inequality in (1.4.9), we have for all $\alpha \in \mathcal{A}(t, z)$:

$$X_s \leq \bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}, \quad t \leq s \leq T, \ a.s. \quad (1.4.13)$$

$$\leq \bar{L}(Z_s^{0,t,z}), \quad t \leq s \leq T, \ a.s. \quad (1.4.14)$$

By definition of L and using (1.4.13), we have :

$$\begin{aligned} 0 \leq L(Z_s) &\leq \max\left(\bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}(1 - \lambda Y_s e^{-\lambda Y_s}), \bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}\right) \\ &\leq \bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}\left(1 - \frac{1}{e}\right), \quad t \leq s \leq T, \ a.s. \end{aligned}$$

since the function $y \mapsto \lambda y e^{-\lambda y}$ is upper bounded by $1/e$. We then deduce

$$P_s \leq \frac{\lambda e}{e-1}\bar{L}(Z_s^{0,t,z}), \quad t \leq s \leq T, \ a.s. \quad (1.4.15)$$

and so (1.4.11) from the arbitrariness of α. By recalling that $X_s + P_s/\lambda \geq 0$ and using (1.4.15), we get

$$-\frac{e}{e-1}\bar{L}(Z_s^{0,t,z}) \quad \leq \quad X_s, \quad t \leq s \leq T, \ a.s.$$

By combining with (1.4.14) and from the arbitrariness of α, we obtain (1.4.10). □

As a third direct corollary, we state that the number of intervention times is finite. More precisely, we have the following result :

Corollary 1.4.4 *For any $(t,z) \in [0,T] \times \bar{\mathcal{S}}$, $\alpha = (\tau_n, \zeta_n) \in \mathcal{A}(t,z)$, the number of intervention times strictly between t and T is finite a.s. :*

$$\begin{aligned} N_t(\alpha) &:= \operatorname{Card}\{n \ : t < \tau_n < T\} \\ &\leq \frac{1}{k}\left[\bar{L}(Z_t) - \bar{L}(Z_{T^-}) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s\right] < \infty \ a.s. \end{aligned} \tag{1.4.16}$$

Proof. Fix some $(t,z) \in [0,T] \times \bar{\mathcal{S}}$ and $\alpha \in \mathcal{A}(t,z)$, and consider $Z_s = (X_s, Y_s, P_s)$, $t \leq s \leq T$, the associated controlled state process. By applying Itô's formula to $\bar{L}(Z_s) = X_s + P_s/\lambda$ between t and T, we have :

$$\begin{aligned} 0 \leq \bar{L}(Z_{T^-}) &= \bar{L}(Z_t) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s + \sum_{t<s<T} \Delta L(Z_s) \\ &\leq \bar{L}(Z_t) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s - kN_t(\alpha), \end{aligned}$$

by (1.4.5). We deduce the required result :

$$N_t(\alpha) \quad \leq \quad \frac{1}{k}\left[\bar{L}(Z_t) - \bar{L}(Z_{T^-}) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s\right] \ < \ \infty \quad a.s.$$

□

1.4.2 Bound on the value function

We can now give a sharp upper bound on the value function.

Proposition 1.4.1 *For all $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$, we have*

$$\sup_{\alpha \in \mathcal{A}(t,z)} U_L(Z_T) \quad \leq \quad U\left(\tilde{L}\left(Z_T^{0,t,z}\right)\right) \quad \in \ L^1(\mathbb{P}). \tag{1.4.17}$$

In particular, the family $\{U_L(Z_T), \alpha \in \mathcal{A}(t,z)\}$ is uniformly integrable and we have

$$v(t,z) \quad \leq \quad v_0(t,z) \ := \ \mathbb{E}\left[U\left(\tilde{L}\left(Z_T^{0,t,z}\right)\right)\right], \ \ (t,z) \in [0,T] \times \bar{\mathcal{S}}, \tag{1.4.18}$$

with

$$v_0(t,z) \quad \leq \quad Ke^{\rho(T-t)}\tilde{L}(z)^{\gamma}, \tag{1.4.19}$$

where ρ is a positive constant s.t.

$$\rho \;>\; \frac{\gamma}{1-\gamma}\frac{b^2+r^2+\sigma^2 r(1-\gamma)}{\sigma^2}. \tag{1.4.20}$$

Proof. Fix $(t,z)\in[0,T]\times\bar{\mathcal{S}}$ and consider for some arbitrary $\alpha\in\mathcal{A}(t,z)$, the process $\tilde{L}(Z_s)$, $t\le s\le T$, which is nonnegative by (1.4.3). By (1.4.8), we have :

$$L(Z_s) \;\le\; \tilde{L}(Z_s) \;\le\; \tilde{L}(Z_s^{0,t,z}) \;=\; X_s^{0,t,x}+\frac{P_s^{0,t,p}}{\lambda}(1-e^{-\lambda y}),\quad t\le s\le T. \tag{1.4.21}$$

From the arbitrariness of α and the nondecreasing property of U, we get the inequality in (1.4.17). From the growth condition (1.2.10) on the nonnegative function U and since clearly $|X_T^{0,t,x}|^\gamma$ and $(P_T^{0,t,p})^\gamma$ are integrable, i.e. in $L^1(\mathbb{P})$, we have $U\left(X_T^{0,t,x}+\frac{P_T^{0,t,p}}{\lambda}(1-e^{-\lambda y})\right)$ $\in L^1(\mathbb{P})$. This clearly implies (1.4.18).

Consider now the nonnegative function :

$$\varphi(t,z) \;=\; e^{\rho(T-t)}\tilde{L}(z)^\gamma \;=\; e^{\rho(T-t)}\left(x+\frac{p}{\lambda}(1-e^{-\lambda y})\right)^\gamma$$

and notice that φ is smooth C^2 on $[0,T]\times(\bar{\mathcal{S}}\setminus D_0)$. We claim that for ρ large enough, the function φ satisfies :

$$-\frac{\partial\varphi}{\partial t}(t,z)-\mathcal{L}\varphi(t,z) \;\ge\; 0,\quad \forall\,(t,z)\in[0,T]\times\bar{\mathcal{S}}\setminus D_0. \tag{1.4.22}$$

Indeed, a straightforward calculation shows that for all $t\in[0,T)$, $z=(x,y,p)\in\bar{\mathcal{S}}\setminus D_0$:

$$\begin{aligned}&-\frac{\partial\varphi}{\partial t}(t,z)-\mathcal{L}\varphi(t,z)\\ &= e^{\rho(T-t)}\tilde{L}(z)^{\gamma-2}\left[Ax^2+B\left(\frac{p}{\lambda}(1-e^{-\lambda y})\right)^2+2Cx\frac{p}{\lambda}(1-e^{-\lambda y})\right],\end{aligned} \tag{1.4.23}$$

where

$$A \;=\; \rho-r\gamma,\qquad B \;=\; \rho-b\gamma+\frac{1}{2}\sigma^2\gamma(1-\gamma),\quad C \;=\; \rho-\frac{(b+r)\gamma}{2}.$$

Hence, (1.4.22) is satisfied whenever $A>0$ and $BC-A^2>0$, which is the case for ρ larger than the constant in the r.h.s. of (1.4.20).

Fix some $(t,z)\in[0,T)\times\bar{\mathcal{S}}$. If $z=(0,0,p)$ then we clearly have $v_0(t,z)=U(0)$ and so inequality (1.4.19) follows from $U(0)\le K_1$ (see (1.2.10)). Consider now the case where $z\in\bar{\mathcal{S}}\setminus D_0$ and notice that the process $Z_s^{0,t,z}=(X_s^{0,t,x},y,P_s^{0,t,p})$ never reaches $\{(0,0)\}\times\mathbb{R}_+^*$. Consider the stopping time

$$T_R \;=\; \inf\left\{s\ge t\;:|Z_s^{0,t,z}|>R\right\}\wedge T$$

so that the stopped process $(Z_{s\wedge T_R}^{0,t,z})_{t\le s\le T}$ stays in the bounded set $\{z=(x,y,p)\in\bar{\mathcal{S}}\setminus D_0\;:\;|z|\le R\}$ on which $\varphi(t,.)$ is smooth C^2 and its derivative in p, $\frac{\partial\varphi}{\partial p}$ is bounded. By applying Itô's formula to $\varphi(s,Z_s^{0,t,z})$ between $s=t$ and $s=T_R$, we have :

$$\varphi(T_R,Z_{T_R}^{0,t,z}) \;=\; \varphi(t,z)+\int_t^{T_R}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(s,Z_s^{0,t,z})ds+\int_t^{T_R}\frac{\partial\varphi}{\partial p}(s,Z_s^{0,t,z})\sigma P_s^{0,t,p}dW_s.$$

Since the integrand in the stochastic integral is bounded, we get by taking expectation in the last relation :

$$\mathbb{E}[\varphi(T_R, Z_{T_R}^{0,t,z})] \;=\; \varphi(t,z) + \mathbb{E}\left[\int_t^{T_R}\left(\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi\right)(s, Z_s^{0,t,z})ds\right] \;\leq\; \varphi(t,z),$$

where we used in the last inequality (1.4.22). Now, for almost $\omega \in \Omega$, for R large enough ($\geq \bar{R}(\omega)$), we have $T_R = T$ so that $\varphi(T_R, Z_{T_R})$ converges a.s. to $\varphi(T, Z_T)$. By Fatou's lemma, we deduce that $\mathbb{E}[\varphi(T, Z_T)] \leq \varphi(t,z)$. Since $\varphi(T,z) = \tilde{L}(z)^\gamma$, this yields

$$\mathbb{E}\left[\tilde{L}\left(Z_T^{0,t,z}\right)^\gamma\right] \;\leq\; \varphi(t,z). \tag{1.4.24}$$

Finally, by the growth condition (1.2.10), this proves the required upper bound on the value function v. □

Remark 1.4.2 The upper bound of the last proposition shows that the value function lies in the set of functions satisfying the growth condition :

$$\mathcal{G}_\gamma([0,T]\times\bar{\mathcal{S}}) \;=\; \left\{u : [0,T]\times\bar{\mathcal{S}} \longrightarrow \mathbb{R}, \quad \sup_{[0,T]\times\bar{\mathcal{S}}} \frac{|u(t,z)|}{1+\left(x+\frac{p}{\lambda}\right)^\gamma} < \infty\right\}.$$

Remark 1.4.3 The upper bound (1.4.18) is sharp in the sense that when λ goes to zero (no price impact), we find the usual Merton bound :

$$v(t,z) \;\leq\; \mathbb{E}[U(X_T^{0,t,x} + yP_T^{0,t,p})] \;\leq\; Ke^{\rho(T-t)}(x+py)^\gamma.$$

As a corollary, we can explicit the value function on the hyperplane of $\bar{\mathcal{S}}$:

$$\bar{\mathcal{S}}^y \;=\; \mathbb{R}_+ \times \{0\} \times \mathbb{R}_+^* \;\subset\; \bar{\mathcal{S}},$$

where the agent does not hold any stock shares.

Corollary 1.4.5 *For any $t \in [0,T)$, $z = (x,0,p) \in \bar{\mathcal{S}}^y$, the investor optimally does not transact during $[t,T]$, i.e.*

$$v(t,z) \;=\; \mathbb{E}\left[U\left(X_T^{0,t,x}\right)\right] \;=\; U\left(xe^{r(T-t)}\right).$$

Proof. For $z = (x,0,p) \in \bar{\mathcal{S}}^y$, let us consider the no impulse control strategy starting from z at t which leads at the terminal date to a net wealth $L(Z_T^{0,t,z}) = X_T^{0,t,x} = xe^{r(T-t)}$. We then have $v(t,z) \geq \mathbb{E}[U(X_T^{0,t,x})] = U(xe^{r(T-t)})$. On the other hand, we have from (1.4.18) : $v(t,z) \leq v_0(t,z) = \mathbb{E}[U(X_T^{0,t,x})]$. This proves the required result. □

1.4.3 Boundary properties

We now turn to the behavior of the value function on the boundary of the solvency region. The situation is more complex than in models with proportional transaction costs where the boundary of the solvency region is an absorbing barrier and all transactions are stopped. Here, the behavior depends on which part of the boundary is the state, as showed in the following proposition.

Proposition 1.4.2 *1) We have*

$$v \;=\; \mathcal{H}v \quad on \;\; [0,T) \times (\partial_\ell^- \mathcal{S} \setminus D_k \cup \partial_\ell^+ \mathcal{S}) \tag{1.4.25}$$

and

$$\mathcal{H}v \;=\; 0 \quad on \;\; [0,T) \times (\partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}) \cap \mathcal{N}_\ell. \tag{1.4.26}$$

2) We have

$$v \;>\; \mathcal{H}v \quad on \;\; [0,T) \times \partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S} \cup D_k. \tag{1.4.27}$$

and

$$v \;=\; 0 \quad on \;\; [0,T) \times D_0, \tag{1.4.28}$$

$$v(t,z) \;=\; U(ke^{r(T-t)}), \quad (t,z) \in [0,T) \in D_k. \tag{1.4.29}$$

Proof. 1. a) Fix some $(t,z) \in [0,T) \times (\partial_\ell^- \mathcal{S} \setminus D_k \cup \partial_\ell^+ \mathcal{S})$ and consider an arbitrary $\alpha = (\tau_n, \zeta_n)_{n \geq 1} \in \mathcal{A}(t,z)$. We claim that $\tau_1 = t$ a.s. i.e. one has to transact immediately at time t in order to satisfy the solvency constraint.

$\star$ Consider first the case where $z \in \partial_\ell^- \mathcal{S} \setminus D_k$. Then on $[t, \tau_1]$, $X_s = xe^{r(s-t)}$, $Y_s = y < 0$, $P_s = pP_s^0$, and so $L(Z_s) = L_0(Z_s^{0,t,z})$. Hence, by integrating between t and τ_1, we get :

$$0 \;\leq\; e^{-r(\tau_1 - t)} L_0(Z_{\tau_1}^{0,t,z}) \;=\; \int_t^{\tau_1} e^{-r(u-t)} P_u^0 y e^{-\lambda y} \left[(b-r)du + \sigma dW_u \right]. \tag{1.4.30}$$

By Girsanov's theorem, one can define a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $\hat{W}_s = W_s + (b-r)s/\sigma$ is a Brownian motion. Under this measure, the stochastic integral $\int_t^{\tau_1} e^{-r(u-t)} P_u^0 y e^{-\lambda y} \sigma d\hat{W}_u$ has zero expectation from which we deduce with (1.4.30) that

$$\int_t^{\tau_1} e^{-r(u-t)} P_u^0 y e^{-\lambda y} \sigma d\hat{W}_u \;=\; 0 \quad a.s.$$

Since $y \neq 0$ and $P_s^0 > 0$ a.s., this implies $\tau_1 = t$ a.s.

$\star$ Consider the case where $z \in \partial_\ell^+ \mathcal{S}$. Then on $[t, \tau_1]$, $X_s = xe^{r(s-t)} < 0$, $Y_s = y$, $P_s = pP_s^0$, and so $L(Z_s) = L_0(Z_s)$. By the same argument as above, we deduce $\tau_1 = t$. Applying the dynamic programming principle (1.2.12) for $\tau = \tau_1$, we clearly deduce (1.4.25).

b) Fix some $(t, z = (x, y, p)) \in [0, T) \times (\partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}) \cap \mathcal{N}_\ell$. Then, from Lemma 1.4.1, $\mathcal{C}(z) = \{-y\}$ and so $\mathcal{H}v(t, z) = v(t, \Gamma(z, -y)) = v(t, 0, 0, p)$. Now, from Corollary 1.4.5, we have for all $z_0 = (0, 0, p) \in D_0$, $v(t, z_0) = U(0) = 0$, which proves (1.4.28) and so (1.4.26).

2. Fix some $t \in [0, T)$ and $z \in \partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S}$. Then by Lemma 1.4.1, $\mathcal{C}(z) = \emptyset$, hence $\mathcal{H}v(t, z) = -\infty$ and so (1.4.27) is trivial. For $z = (k, 0, p) \in D_k$, we have by Lemma 1.4.1, $\mathcal{C}(z) = \{0\}$ and so $\mathcal{H}v(t, z) = v(t, \Gamma(z, 0)) = v(t, 0, 0, p) = 0$ by (1.4.28). Therefore, from Corollary 1.4.5, we have for $z = (k, 0, p) \in D_k : v(t, z) = U(ke^{r(T-t)}) > 0 = \mathcal{H}v(t, z)$. □

Remark 1.4.4 The last proposition and its proof means that when the state reaches $\partial_\ell^- \mathcal{S} \setminus D_k \cup \partial_\ell^+ \mathcal{S}$, one has to transact immediately since the no transaction strategy is not admissible. Moreover, if one is in $(\partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}) \cap \mathcal{N}_\ell$, one jumps directly to D_0 where all transactions are stopped. On the other hand, if the state is in $\partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S} \cup D_k$, one should do not transact : admissible transaction does not exist on $\partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S}$ while the only zero admissible transaction on D_k is suboptimal with respect to the no transaction control. In the remaining part $\partial_2^x \mathcal{S}$ of the boundary, both decisions, transaction and no-transaction, are admissible : we only know that one of these decisions should be chosen optimally but we are not able to be explicit about which one is optimal. A representation of the behavior of the optimal strategy on the boundary of the solvency region is depicted in Figures 2 and 3.

The next result states the continuity of the value function on the part D_0 of the solvency boundary, as a direct consequence of (1.4.18) and (1.4.28).

Corollary 1.4.6 *The value function v is continuous on $[0, T) \times D_0$:*

$$\lim_{(t',z') \to (t,z)} v(t', z') \;=\; v(t, z) \;=\; 0, \quad \forall (t, z) \in [0, T) \times D_0.$$

Remark 1.4.5 Notice that except on D_0, the value function is in general discontinuous on the boundary of the solvency region. More precisely, for any $t \in [0, T)$, $z \in D_k$, we have from (1.4.25)-(1.4.26) :

$$\lim_{\substack{z' \to z \\ z' \in \partial_\ell^- \mathcal{S} \setminus D_k}} v(t, z') \;=\; 0,$$

while from Corollary 1.4.5 :

$$\lim_{\substack{z' \to z \\ z' \in \bar{S}^y}} v(t, z') \;=\; U(ke^{r(T-t)}).$$

This shows that v is discontinuous on $[0, T) \times D_k$. Similarly, one can show that v is discontinuous on $[0, T) \times (\partial_1^x \mathcal{S} \cap \partial_\ell^+ \mathcal{S})$.

1.4.4 Terminal condition

We end this section by determining the right terminal condition of the value function. We set

$$v^*(T,z) := \limsup_{\substack{(t,z')\to(T,z)\\ t<T,\, z'\in\mathcal{S}}} v(t,z'), \qquad v_*(T,z) := \liminf_{\substack{(t,z')\to(T,z)\\ t<T,\, z'\in\mathcal{S}}} v(t,z')$$

Proposition 1.4.3 *We have*

$$v_*(T,z) \;=\; v^*(T,z) \;=\; \bar U(z), \quad \forall z\in\bar{\mathcal{S}},$$

where

$$\bar U(z) \;:=\; \max\left[U_L(z), \mathcal{H}U_L(z)\right].$$

Proof. 1) Fix some $z\in\bar{\mathcal{S}}$ and consider some sequence $(t_m,z_m)_m\in[0,T)\times\mathcal{S}$ converging to (T,z) and s.t. $\lim_m v(t_m,z_m) = v_*(T,z)$. By taking the no impulse control strategy on $[t_m,T]$, we have

$$v(t_m,z_m) \;\geq\; \mathbb{E}\left[U_L(Z_T^{0,t_m,z_m})\right].$$

Since Z_T^{0,t_m,z_m} converges a.s. to z when m goes to infinity by continuity of the diffusion $Z^{0,t,z}$ in its initial conditions (t,z), we deduce by Fatou's lemma that :

$$v_*(T,z) \;\geq\; U_L(z). \tag{1.4.31}$$

Take now some arbitrary $\zeta\in\mathcal{C}(z)$. Consider first the case where $L(\Gamma(z,\zeta))>0$. We claim that for m large enough, $\zeta\in\mathcal{C}(z_m)$. Indeed,

$\star$ suppose that $\zeta\neq -y$. Then, by continuity of the function $z'\mapsto L(\Gamma(z',\zeta))$ on $\{z'=(x',y',p')\,:\,y'\neq\zeta\}$, we deduce that $L(\Gamma(z_m,\zeta))$ converges to $L(\Gamma(z,\zeta))>0$ and so for m large enough, $\zeta\in\mathcal{C}(z_m)$.

$\star$ Suppose that $\zeta=-y$, i.e. $L(\Gamma(z,\zeta)) = x+\ell(y,p)-k>0$. Notice that

$$\begin{aligned} L(\Gamma(z_m,\zeta)) \;&=\; \max\left[L_0(z_m)-k+pg(-y,y_m), x_m+ye^{-\lambda y}p_m-k\right]1_{y_m-y\geq 0}\\ &\quad+\; L_0(z_m)1_{y_m-y<0}.\end{aligned}$$

We then see that $\liminf_{m\to\infty} L(\Gamma(z_m,\zeta))\geq L(\Gamma(z,\zeta))$, and so for m large enough, $\zeta\in\mathcal{C}(z_m)$.

One may then consider the admissible control with immediate impulse at t_m with size ζ and no other impulse until T so that the associated state process is $Z^{t_m,z_m} = Z^{0,t_m,\Gamma(z_m,\zeta)}$ and thus

$$v(t_m,z_m) \;\geq\; \mathbb{E}\left[U_L\left(Z_T^{0,t_m,\Gamma(z_m,\zeta)}\right)\right].$$

Sending m to infinity, we obtain :

$$v_*(T,z) \quad \geq \quad U_L(\Gamma(z,\zeta)), \tag{1.4.32}$$

for all ζ in $\mathcal{C}(z)$ s.t. $L(\Gamma(z,\zeta)) > 0$. This last inequality (1.4.32) holds obviously true when $L(\Gamma(z,\zeta)) = 0$ since in this case $U_L(\Gamma(z,\zeta)) = 0 \leq v_*(T,z)$. By combining with (1.4.31), we get $v_*(T,z) \geq \bar{U}(z)$.

2) Fix some $z \in \bar{\mathcal{S}}$ and consider some sequence $(t_m, z_m)_m \in [0,T) \times \mathcal{S}$ converging to (T,z) and s.t. $\lim_m v(t_m,z_m) = v^*(T,z)$. For any m, one can find $\hat{\alpha}^m = (\hat{\tau}_n^m, \hat{\zeta}_n^m)_n \in \mathcal{A}(t_m,z_m)$ s.t.

$$v(t_m,z_m) \quad \leq \quad \mathbb{E}\left[U_L(\hat{Z}_T^m)\right] + \frac{1}{m} \tag{1.4.33}$$

where $\hat{Z}^m = (\hat{X}^m, \hat{Y}^m, \hat{P}^m)$ denotes the state process controlled by $\hat{\alpha}^m$ and given in T by :

$$\begin{aligned} \hat{Z}_T^m \;=\; \hat{Z}_{T^-}^m &= z_m + \int_{t_m}^T B(\hat{Z}_s^m)ds + \int_{t_m}^T \Sigma(\hat{Z}_s^m)dW_s + \sum_{t_m \leq u < T} \Delta \hat{Z}_s^m \\ &= z_m + (\Gamma(z_m, \zeta_1^m) - z_m)\, 1_{\tau_1^m = t_m} + R_T^m \end{aligned} \tag{1.4.34}$$

with $B(z) = (rx, 0, bp)$ and $\Sigma(z) = (0,0,\sigma p)$ and

$$R_T^m \quad = \quad \int_{t_m}^T B(\hat{Z}_s^m)ds + \int_{t_m}^T \Sigma(\hat{Z}_s^m)dW_s + \sum_{t_m < s < T} \Delta \hat{Z}_s^m. \tag{1.4.35}$$

We rewrite (1.4.33) as

$$\begin{aligned} v(t_m,z_m) \quad \leq \quad & \mathbb{E}\left[\{U_L(\Gamma(z_m,\zeta_1^m) + R_T^m) - U_L(z_m + R_T^m)\}\, 1_{\tau_1^m = t_m} \right. \\ & \left. + U_L(z_m + R_T^m)\right] + \frac{1}{m} \end{aligned} \tag{1.4.36}$$

We claim that R_T^m converges a.s. to 0 as m goes to infinity. Indeed, from the uniform bounds (1.4.10)-(1.4.11), we have

$$\begin{aligned} |B(\hat{Z}_s^m)| + |\Sigma(\hat{Z}_s^m)| \quad &\leq \quad (r + (b+\sigma)\lambda)\frac{e}{e-1}\, L(Z_s^{0,t,z_m}) \\ &\leq \quad \text{Cte } L(Z_s^{0,t,z}), \quad t_m \leq s \leq T, \; a.s., \end{aligned}$$

for some positive Cte independent of m. We then deduce that the Lebesgue and stochastic integral in (1.4.35) converge a.s. to zero as m goes to infinity, i.e. t_m goes to T. On the other hand, by same argument as in Remark 1.4.1, we see that for each $t_m < s < T$, the jump ΔZ_s^m is uniformly bounded in m. Moreover, by (1.4.16), we have

$$N_{t_m}(\hat{\alpha}^m) \leq \frac{1}{k}\left[\bar{L}(\hat{Z}_{t_m}^m) - \bar{L}(\hat{Z}_{T^-}^m) + \int_{t_m}^T \left(r\hat{X}_s^m + \frac{\hat{P}_s^m}{\lambda}\right)ds + \int_{t_m}^T \frac{\sigma}{\lambda}\hat{P}_s^m dW_s\right]. \tag{1.4.37}$$

Similarly as above, by the uniform bounds in (1.4.10)-(1.4.11), the integrals in (1.4.37) converge to zero as m goes to infinity. From the left-continuity of the state process $\hat{Z}^m$ and the continuity of $\bar{L}$, we deduce that $\bar{L}(\hat{Z}^m_{t^m}) - \bar{L}(Z^m_{T_-})$ converge to zero as m goes to infinity. Therefore, $\sum_{t_m<s<T}\Delta\hat{Z}^m_s$ goes to zero as m goes to infinity, which proves the required zero convergence of R^m_T.

By Remark 1.4.1, the sequence of jump size $(\zeta^m_1)_m$ is bounded, and up to a subsequence, converges, as m goes to infinity, to some $\zeta \in \mathcal{C}(z)$. Moreover, it is easy to check that the family $\{U(X^{0,t_m,x_m}_T + \frac{P^{0,t_m,p_m}_T}{\lambda}(1-e^{-\lambda y_m})), m \geq 1\}$ is uniformly integrable so that from (1.4.17), the family $\{U_L(\hat{Z}^m_T), m \geq 1\}$ is also uniformly integrable. Therefore, we can send m to infinity into (1.4.33) (or (1.4.36)) by the dominated convergence theorem and get :

$$
\begin{aligned}
v^*(T,z) &\leq \mathbb{E}\left[\{U_L(\Gamma(z,\zeta)) - U_L(z)\}\limsup_{m\to\infty} 1_{\tau^m_1=t_m} + U_L(z)\right] \\
&\leq \max\left\{U_L(z), \sup_{\zeta\in\mathcal{C}(z)} U_L(\Gamma(z,\zeta))\right\}.
\end{aligned}
$$

By completing with (1.4.31), this proves $v_*(T,z) = v^*(T,z) = \bar{U}(z)$. □

Remark 1.4.6 The previous result shows in particular that the value function is discontinuous on T. Indeed, recalling that we do not allow any impulse transaction at T, we have $v(T,z) = U_L(z)$ for all $z \in \bar{\mathcal{S}}$. Moreover, by Proposition 1.4.3, we have $v(T^-,z) = \bar{U}(z)$, hence $v(.,z)$ is discontinuous on T for all $z \in \{z \in \bar{\mathcal{S}} \ : \mathcal{H}U_L(z) > U_L(z)\} \neq \emptyset$.

1.5 Viscosity characterization

In this section, we intend to provide a rigorous characterization of the value function by means of (constrained) viscosity solution to the quasi-variational inequality :

$$
\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v, v - \mathcal{H}v\right] = 0, \tag{1.5.1}
$$

together with appropriate boundary and terminal conditions.

As mentioned previously, the value function is not known to be continuous a priori and so we shall work with the notion of discontinuous viscosity solutions. For a locally bounded function u on $[0,T)\times\bar{\mathcal{S}}$ (which is the case of the value function v), we denote by u_* (resp. u^*) the lower semi-continuous (lsc) (resp. upper semi-continuous (usc)) envelope of u. We recall that in general, $u_* \leq u \leq u^*$, and that u is lsc iff $u = u_*$, u is usc iff $u = u^*$, and u is continuous iff $u_* = u^*$ $(= u)$. We denote by $LSC([0,T)\times\bar{\mathcal{S}})$ (resp. $USC([0,T)\times\bar{\mathcal{S}})$) the set of lsc (resp. usc) functions on $[0,T)\times\bar{\mathcal{S}}$.

We work with the suitable notion of constrained viscosity solutions, introduced in [62] for first-order equations, for taking into account boundary conditions arising in state constraints. The use of constrained viscosity solutions was initiated in [67] for stochastic control problems arising in optimal investment problems. The definition is given as follows :

Definition 1.5.1 *(i) Let $\mathcal{O} \subset \bar{\mathcal{S}}$. A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a viscosity subsolution (resp. supersolution) of* (1.5.1) *in* $[0,T) \times \mathcal{O}$ *if for all* $(\bar{t}, \bar{z}) \in [0,T) \times \mathcal{O}$ *and* $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}})$ *s.t.* $(u^* - \varphi)(\bar{t}, \bar{z}) = 0$ *(resp.* $(u_* - \varphi)(\bar{t}, \bar{z}) = 0$*) and* $(\bar{t}, \bar{z})$ *is a maximum of* $u^* - \varphi$ *(resp. minimum of* $u_* - \varphi$*) on* $[0,T) \times \mathcal{O}$*, we have*

$$
\begin{aligned}
\min\left[-\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{z}) - \mathcal{L}\varphi(\bar{t}, \bar{z}), u^*(\bar{t}, \bar{z}) - \mathcal{H}u^*(\bar{t}, \bar{z})\right] &\leq 0 && (1.5.2)\\
(\textit{ resp.} &\geq 0). && (1.5.3)
\end{aligned}
$$

(ii) A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a constrained viscosity solution of (1.5.1) *in* $[0,T) \times \mathcal{S}$ *if u is a viscosity subsolution of* (1.5.1) *in* $[0,T) \times \bar{\mathcal{S}}$ *and a viscosity supersolution of* (1.5.1) *in* $[0,T) \times \mathcal{S}$.

Remark 1.5.1 There is an equivalent formulation of viscosity solutions, which is useful for proving uniqueness results, see [19] :

(i) Let $\mathcal{O} \subset \bar{\mathcal{S}}$. A function $u \in USC([0,T) \times \bar{\mathcal{S}})$ is a viscosity subsolution (resp. supersolution) of (1.5.1) in $[0,T) \times \mathcal{O}$ if

$$
\begin{aligned}
\min\left[-q_0 - rxq_1 - bpq_3 - \frac{1}{2}\sigma^2 p^2 M_{33}, u(t,z) - \mathcal{H}u(t,z)\right] &\leq 0 && (1.5.4)\\
(\text{ resp.} &\geq 0) && (1.5.5)
\end{aligned}
$$

for all $(t, z = (x,y,p)) \in [0,T) \times \mathcal{O}$, $(q_0, q = (q_i)_{1 \leq i \leq 3}, M = (M_{ij})_{1 \leq i,j \leq 3}) \in \bar{J}^{2,+}u(t,z)$ (resp. $\bar{J}^{2,-}u(t,z)$).

(ii) A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a constrained viscosity solution to (1.5.1) if u^* satisfies (1.5.4) for all $(t,z) \in [0,T) \times \bar{\mathcal{S}}$, $(q_0, q, M) \in \bar{J}^{2,+}u^*(t,z)$, and u_* satisfies (1.5.5) for all $(t,z) \in [0,T) \times \mathcal{S}$, $(q_0, q, M) \in \bar{J}^{2,-}u_*(t,z)$.

Here $J^{2,+}u(t,z)$ is the parabolic second order superjet defined by :

$$
\begin{aligned}
J^{2,+}u(t,z) \; = \; \Big\{ & (q_0, q, M) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{S}^3 \; : \\
& \limsup_{\substack{(t',z') \to (t,z) \\ (t',z') \in [0,T) \times \mathcal{S}}} \frac{u(t',z') - u(t,z) - q_0(t'-t) - q.(z'-z) - \frac{1}{2}(z'-z).M(z'-z)}{|t'-t| + |z'-z|^2} \leq 0 \Big\},
\end{aligned}
$$

$\mathbb{S}^3$ is the set of symmetric 3×3 matrices, $\bar{J}^{2,+}u(t,z)$ is its closure :

$$
\begin{aligned}
\bar{J}^{2,+}u(t,x) \; = \; \Big\{ (q_0, q, M) \; &= \; \lim_{m \to \infty} (q_0^m, q^m, M^m) \quad \text{with } (q_0^m, q^m, M^m) \in J^{2,+}u(t_m, z_m) \\
\text{and } \; \lim_{m \to \infty} (t_m, z_m, u(t_m, z_m)) \; &= \; (t, z, u(t,z)) \Big\},
\end{aligned}
$$

and $J^{2,-}u(t,x) = -J^{2,+}(-u)(t,x)$, $\bar{J}^{2,-}u(t,x) = -\bar{J}^{2,+}(-u)(t,x)$.

1.5.1 Viscosity property

Our first main result of this section is the following.

Theorem 1.5.1 *The value function v is a constrained viscosity solution to* (1.5.1) *in* $[0,T)\times\mathcal{S}$.

Remark 1.5.2 The state constraint and the boundary conditions is translated through the PDE characterization via the subsolution property, which has to hold true on the whole closed region $\bar{\mathcal{S}}$. This formalizes the property that on the boundary of the solvency region, one of the two possible decisions, immediate impulse transaction or no-transaction, should be chosen optimally.

We need some auxiliary results on the impulse operator $\mathcal{H}$.

Lemma 1.5.1 *Let u be a locally bounded function on $[0,T)\times\bar{\mathcal{S}}$.*
(i) $\mathcal{H}u_* \leq (\mathcal{H}u)_*$. Moreover, if u is lsc then $\mathcal{H}u$ is also lsc.
(ii) $\mathcal{H}u^*$ is usc and $(\mathcal{H}u)^* \leq \mathcal{H}u^*$.

Proof. (i) Let (t_n,z_n) be a sequence in $[0,T)\times\bar{\mathcal{S}}$ converging to (t,z) and s.t. $\mathcal{H}u(t_n,z_n)$ converges to $(\mathcal{H}u)_*(t,z)$. Then, using also the lower-semicontinuity of u_* and the continuity of Γ, we have :

$$\begin{aligned}\mathcal{H}u_*(t,z) &= \sup_{\zeta\in\mathcal{C}(z)} u_*(t,\Gamma(z,\zeta)) \leq \sup_{\zeta\in\mathcal{C}(z)} \liminf_{n\to\infty} u_*(t_n,\Gamma(z_n,\zeta)) \\ &\leq \liminf_{n\to\infty} \sup_{\zeta\in\mathcal{C}(z)} u_*(t_n,\Gamma(z_n,\zeta)) \leq \lim_{n\to\infty} \mathcal{H}u(t_n,z_n) = (\mathcal{H}u)_*(t,z).\end{aligned}$$

Suppose now that u is lsc and let $(t,z)\in[0,T)\times\bar{\mathcal{S}}$ and let $(t_n,z_n)_{n\geq1}$ be a sequence in $[0,T)\times\bar{\mathcal{S}}$ converging to (t,z) (as n goes to infinity). By definition of the lsc envelope $(\mathcal{H}u)_*$, we then have :

$$\mathcal{H}u(t,z) = \mathcal{H}u_*(t,z) \leq (\mathcal{H}u)_*(t,z) \leq \liminf_{n\to\infty} \mathcal{H}u(t_n,z_n),$$

which shows the lower-semicontinuity of $\mathcal{H}u$.
(ii) Fix some $(t,z)\in[0,T)\times\bar{\mathcal{S}}$ and let $(t_n,z_n)_{n\geq1}$ be a sequence in $[0,T)\times\bar{\mathcal{S}}$ converging to (t,z) (as n goes to infinity). Since u^* is usc, Γ is continuous, and $\mathcal{C}(z_n)$ is compact for each $n\geq1$, there exists a sequence $(\hat{\zeta}_n)_{n\geq1}$ with $\hat{\zeta}_n\in\mathcal{C}(z_n)$ such that :

$$\mathcal{H}u^*(t_n,z_n) = u^*(t_n,\Gamma(z_n,\hat{\zeta}_n)), \quad \forall n\geq1.$$

By Remark 1.4.1, the sequence $(\hat{\zeta}_n)_{n\geq1}$ converges, up to a subsequence, to some $\hat{\zeta}\in\mathcal{C}(z)$. Therefore, we get :

$$\mathcal{H}u^*(t,z) \geq u^*(t,\Gamma(z,\hat{\zeta})) \geq \limsup_{n\to\infty} u^*(t_n,\Gamma(z_n,\hat{\zeta}_n)) = \limsup_{n\to\infty} \mathcal{H}u^*(t_n,z_n),$$

which proves that $\mathcal{H}u^*$ is usc.

On the other hand, fix some $(t,z) \in [0,T] \times \bar{\mathcal{S}}$ and let $(t_n,z_n)_{n\geq 1}$ be a sequence in $[0,T] \times \bar{\mathcal{S}}$ converging to (t,z) and s.t. $\mathcal{H}u(t_n,z_n)$ converges to $(\mathcal{H}u)^*(t,z)$. Then, we have

$$(\mathcal{H}u)^*(t,z) \;=\; \lim_{n\to\infty} \mathcal{H}u(t_n,z_n) \;\;\leq\;\; \limsup_{n\to\infty} \mathcal{H}u^*(t_n,z_n) \;\leq\; \mathcal{H}u^*(t,z),$$

which shows that $(\mathcal{H}u)^* \leq \mathcal{H}u^*$. □

We may then prove by standard arguments, using DPP (1.2.13), the supersolution property.

Proof of supersolution property on $[0,T) \times \mathcal{S}$.

First, for any $(t,z) \in [0,T] \times \bar{\mathcal{S}}$, we see, as a consequence of (DPP) (1.2.13) applied to $\tau = t$, and by choosing any admissible control $\alpha \in \mathcal{A}(t,z)$ with immediate impulse at t of arbitrary size $\zeta \in \mathcal{C}(z)$, that $v(t,z) \geq \mathcal{H}v(t,z)$. Now, let $(\bar{t},\bar{z}) \in [0,T) \times \mathcal{S}$ and $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}})$ s.t. $v_*(\bar{t},\bar{z}) = \varphi(\bar{t},\bar{z})$ and $\varphi \leq v_*$ on $[0,T) \times \mathcal{S}$. Since $v \geq \mathcal{H}v$ on $[0,T] \times \bar{\mathcal{S}}$, we obtain by combining with Lemma 1.5.1 (i) that $\mathcal{H}v_*(\bar{t},\bar{z}) \leq (\mathcal{H}v)_*(\bar{t},\bar{z}) \leq v_*(\bar{t},\bar{z})$, and so it remains to show that

$$-\frac{\partial \varphi}{\partial t}(\bar{t},\bar{z}) - \mathcal{L}\varphi(\bar{t},\bar{z}) \;\;\geq\;\; 0. \tag{1.5.6}$$

From the definition of v_*, there exists a sequence $(t_m,z_m)_{m\geq 1} \in [0,T) \times \mathcal{S}$ s.t. (t_m,z_m) and $v(t_m,z_m)$ converge respectively to $(\bar{t},\bar{z})$ and $v_*(\bar{t},\bar{z})$ as m goes to infinity. By continuity of φ, we also have that $\gamma_m := v(t_m,z_m) - \varphi(t_m,z_m)$ converges to 0 as m goes to infinity. Since $(\bar{t},\bar{z}) \in [0,T) \times \mathcal{S}$, there exists $\eta > 0$ s.t. for m large enough, $t_m < T$ and $B(z_m,\eta/2)$ $\subset B(\bar{z},\eta) := \{|z - \bar{z}| < \eta\} \subset \mathcal{S}$. Let us then consider the admissible control in $\mathcal{A}(t_m,z_m)$ with no impulse until the first exit time τ_m before T of the associated state process $Z_s =$ Z_s^{0,t_m,z_m} from $B(z_m,\eta/2)$:

$$\tau_m \;\;=\;\; \inf\left\{s \geq t_m \;: |Z_s^{0,t_m,z_m} - z_m| \geq \eta/2\right\} \wedge T.$$

Consider also a strictly positive sequence $(h_m)_m$ s.t. h_m and γ_m/h_m converge to zero as m goes to infinity. By using the dynamic programming principle (1.2.13) for $v(t_m,z_m)$ and $\hat{\tau}_m$ $:= \tau_m \wedge (t_m + h_m)$, we get :

$$v(t_m,z_m) \;=\; \gamma_m + \varphi(t_m,z_m) \;\;\geq\;\; \mathbb{E}[v(\hat{\tau}_m, Z_{\hat{\tau}_m}^{0,t_m,z_m})] \;\geq\; E[\varphi(\hat{\tau}_m, Z_{\hat{\tau}_m}^{0,t_m,z_m}),$$

since $\varphi \leq v_* \leq v$ on $[0,T) \times \mathcal{S}$. Now, by applying Itô's formula to $\varphi(s, Z_s^{0,t_m,z_m})$ between t_m and $\hat{\tau}_m$ and noting that the integrand of the stochastic integral term is bounded, we obtain by taking expectation :

$$\frac{\gamma_m}{h_m} + \mathbb{E}\left[\frac{1}{h_m}\int_{t_m}^{\hat{\tau}_m}\left(-\frac{\partial\varphi}{\partial t} - \mathcal{L}\varphi\right)(s, Z_s^{0,t_m,z_m})ds\right] \;\;\geq\;\; 0. \tag{1.5.7}$$

By continuity a.s. of Z_s^{0,t_m,z_m}, we have for m large enough, $\hat{\tau}_m = t_m + h_m$, and so by the mean-value theorem, the random variable inside the expectation in (1.5.7) converges a.s. to $(-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi)(\bar{t}, \bar{z})$ as m goes to infinity. Since this random variable is also bounded by a constant independent of m, we conclude by the dominated convergence theorem and obtain (1.5.6).

We next prove the subsolution property, by using DPP (1.2.14) and contraposition argument.

Proof of subsolution property on $[0, T) \times \bar{\mathcal{S}}$.
Let $(\bar{t}, \bar{z}) \in [0, T) \times \bar{\mathcal{S}}$ and $\varphi \in C^{1,2}([0, T) \times \bar{\mathcal{S}})$ s.t. $v^*(\bar{t}, \bar{z}) = \varphi(\bar{t}, \bar{z})$ and $\varphi \geq v^*$ on $[0, T) \times \bar{\mathcal{S}}$. If $v^*(\bar{t}, \bar{z}) \leq \mathcal{H}v^*(\bar{t}, \bar{z})$ then the subsolution inequality holds trivially. Consider now the case where $v^*(\bar{t}, \bar{z}) > \mathcal{H}v^*(\bar{t}, \bar{z})$ and argue by contradiction by assuming on the contrary that

$$\eta \; := \; -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{z}) - \mathcal{L}\varphi(\bar{t}, \bar{z}) \quad > \quad 0.$$

By continuity of φ and its derivatives, there exists some $\delta_0 > 0$ s.t. $\bar{t} + \delta_0 < T$ and for all $0 < \delta \leq \delta_0$:

$$-\frac{\partial \varphi}{\partial t}(t, z) - \mathcal{L}\varphi(t, z) \quad > \quad \frac{\eta}{2}, \quad \forall \, (t, z) \in ((\bar{t} - \delta)_+, \bar{t} + \delta) \times B(\bar{z}, \delta) \cap \bar{\mathcal{S}}. \qquad (1.5.8)$$

From the definition of v^*, there exists a sequence $(t_m, z_m)_{m \geq 1} \in ((\bar{t} - \delta/2)_+, \bar{t} + \delta/2) \times B(\bar{z}, \delta/2) \cap \bar{\mathcal{S}}$ s.t. (t_m, z_m) and $v(t_m, z_m)$ converge respectively to $(\bar{t}, \bar{z})$ and $v^*(\bar{t}, \bar{z})$ as m goes to infinity. By continuity of φ, we also have that $\gamma_m := v(t_m, z_m) - \varphi(t_m, z_m)$ converges to 0 as m goes to infinity. By the dynamic programming principle (1.2.14), given $m \geq 1$, there exists $\hat{\alpha}^m = (\hat{\tau}_n^m, \hat{\zeta}_n^m)_{n \geq 1}$ s.t. for any stopping time τ valued in $[t_m, T]$, we have

$$v(t_m, z_m) \quad \leq \quad \mathbb{E}[v(\tau, \hat{Z}_\tau^m)] + \frac{1}{m}.$$

Here $\hat{Z}^m$ is the state process, starting from z_m at t_m, and controlled by $\hat{\alpha}^m$. By choosing $\tau = \bar{\tau}^m := \hat{\tau}_1^m \wedge \tau_\delta^m$ where

$$\tau_\delta^m \quad = \quad \inf \Big\{ s \geq t_m \; : \hat{Z}_s^m \notin B(z_m, \delta/2) \Big\} \wedge (t_m + \delta/2)$$

is the first exit time before $t_m + \delta/2$ of $\hat{Z}^m$ from the open ball $B(z_m, \delta/2)$, we then get :

$$\begin{aligned} v(t_m, z_m) \quad &\leq \quad \mathbb{E}[v(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-}) 1_{\tau_\delta^m < \hat{\tau}_1^m}] + \mathbb{E}[v(\bar{\tau}^m, \Gamma(\hat{Z}_{\bar{\tau}^m,-}, \hat{\zeta}_1^m)) 1_{\hat{\tau}_1^m \leq \tau_\delta^m}] + \frac{1}{m} \\ &\leq \quad \mathbb{E}[v(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-}) 1_{\tau_\delta^m < \hat{\tau}_1^m}] + \mathbb{E}[\mathcal{H}v(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-}) 1_{\hat{\tau}_1^m \leq \tau_\delta^m}] + \frac{1}{m}. \qquad (1.5.9) \end{aligned}$$

Now, since $\mathcal{H}v \leq v \leq v^* \leq \varphi$ on $[0, T) \times \bar{\mathcal{S}}$, we obtain :

$$\varphi(t_m, z_m) + \gamma_m \quad \leq \quad \mathbb{E}[\varphi(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-})] + \frac{1}{m}.$$

By applying Itô's formula to $\varphi(s, \hat{Z}_s^m)$ between t_m and $\bar{\tau}_m$, we then get :

$$\gamma_m \quad \leq \quad \mathbb{E}\left[\int_{t_m}^{\bar{\tau}_m} \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi\right)(s, \hat{Z}_s^m) ds\right] + \frac{1}{m} \ \leq \ -\frac{\eta}{2}\mathbb{E}[\bar{\tau}_m - t_m] + \frac{1}{m},$$

from (1.5.8). This implies

$$\lim_{m \to \infty} \mathbb{E}[\bar{\tau}_m] \quad = \quad \bar{t}. \tag{1.5.10}$$

On the other hand, we have by (1.5.9)

$$v(t_m, z_m) \quad \leq \quad \sup_{\substack{|t' - t| < \delta \\ |z' - z| < \delta}} v(t', z')\mathbb{P}[\tau_\delta^m < \hat{\tau}_1^m] + \sup_{\substack{|t' - t| < \delta \\ |z' - z| < \delta}} \mathcal{H}v(t', z')\mathbb{P}[\hat{\tau}_1^m \leq \tau_\delta^m] + \frac{1}{m}.$$

From (1.5.10), we then get by sending m to infinity :

$$v^*(\bar{t}, \bar{z}) \quad \leq \quad \sup_{\substack{|t' - t| < \delta \\ |z' - z| < \delta}} \mathcal{H}v(t', z').$$

Hence, sending δ to zero and by Lemma 1.5.1 (ii), we have

$$v^*(\bar{t}, \bar{z}) \quad \leq \quad \lim_{\delta \downarrow 0} \sup_{\substack{|t' - t| < \delta \\ |z' - z| < \delta}} \mathcal{H}v(t', z') \ = \ (\mathcal{H}v)^*(\bar{t}, \bar{z}) \ \leq \ \mathcal{H}^*v(\bar{t}, \bar{z}),$$

which is the required contradiction.

1.5.2 Comparison principle

We finally turn to uniqueness question. Our next main result is a comparison principle for constrained (discontinuous) viscosity solutions to the quasi-variational inequality (1.5.1). It states that we can compare a viscosity subsolution to (1.5.1) on $[0, T) \times \bar{\mathcal{S}}$ and a viscosity supersolution to (1.5.1) on $[0, T) \times \mathcal{S}$, provided that we can compare them at the terminal date (as usual in parabolic problems) but also on the part D_0 of the solvency boundary.

Theorem 1.5.2 *Suppose $u \in \mathcal{G}_\gamma([0, T] \times \bar{\mathcal{S}}) \cap USC([0, T) \times \bar{\mathcal{S}})$ is a viscosity subsolution to (1.5.1) in $[0, T) \times \bar{\mathcal{S}}$ and $w \in \mathcal{G}_\gamma([0, T] \times \bar{\mathcal{S}}) \cap LSC([0, T) \times \bar{\mathcal{S}})$ is a viscosity supersolution to (1.5.1) in $[0, T) \times \mathcal{S}$ such that :*

$$u(t, z) \quad \leq \quad \liminf_{(t', z') \to (t, z)} w(t', z'), \ \forall \ (t, z) \in [0, T) \times D_0, \tag{1.5.11}$$

$$u(T, z) := \limsup_{\substack{(t, z') \to (T, z) \\ t < T, z' \in \mathcal{S}}} u(t, z') \quad \leq \quad w(T, z) := \liminf_{\substack{(t, z') \to (T, z) \\ t < T, z' \in \mathcal{S}}} w(t, z'), \quad \forall \ z \in \bar{\mathcal{S}}. \tag{1.5.12}$$

Then,

$$u \quad \leq \quad w \quad \text{on} \ \ [0, T] \times \mathcal{S}.$$

Remark 1.5.3 Notice that one cannot hope to derive a comparison principle in the whole closed region $\bar{\mathcal{S}}$ since it would imply the continuity of the value function on $\bar{\mathcal{S}}$, which is not true, see Remark 1.4.5.

In order to deal with the impulse obstacle in the comparison principle, we first produce some suitable perturbation of viscosity supersolutions. This strict viscosity supersolution argument was introduced by [41], and used e.g. in [1] for dealing with gradient constraints in singular control problem.

Lemma 1.5.2 *Let $\gamma' \in (0,1)$ and choose ρ' s.t.*

$$\rho' \quad > \quad \frac{\gamma'}{1-\gamma'} \frac{b^2 + r^2 + \sigma^2 r(1-\gamma')}{\sigma^2} \vee b \vee (\sigma^2 - b)$$

Given $\nu \geq 0$, consider the perturbation smooth function on $[0,T] \times \bar{\mathcal{S}}$:

$$\phi_\nu(t,z) \quad = \quad e^{\rho'(T-t)} \left[\tilde{L}(z)^{\gamma'} + \nu \left(\frac{e^{\lambda y}}{p} + p e^{-\lambda y} \right) \right]. \tag{1.5.13}$$

Let $w \in LSC([0,T) \times \bar{\mathcal{S}})$ be a viscosity supersolution to (1.5.1) in $[0,T) \times \mathcal{S}$. Then for any $m \geq 1$, any compact set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$, the usc function

$$w_m \quad = \quad w + \frac{1}{m}\phi_\nu$$

is a strict viscosity supersolution to (1.5.1) in $[0,T) \times \mathcal{S} \cap \mathcal{K}$: there exists some constant δ (depending on $\mathcal{K}$) s.t.

$$\min \left[-q_0 - rxq_1 - bpq_3 - \frac{1}{2}\sigma^2 p^2 M_{33}, w_m(t,z) - \mathcal{H} w_m(t,z) \right] \quad \geq \quad \frac{\delta}{m}, \tag{1.5.14}$$

for all $(t, z = (x,y,p)) \in [0,T) \times \mathcal{S} \cap \mathcal{K}$, $(q_0, q = (q_i)_{1 \leq i \leq 3}, M = (M_{ij})_{1 \leq i,j \leq 3}) \in \bar{J}^{2,-} w_m(t,z)$. Moreover, for $\gamma \in (0,\gamma')$ and $\nu > 0$, if $w \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$, and u is also a function in $\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$, then for any $t \in [0,T]$, $m \geq 1$,

$$\lim_{|z| \to \infty} (u - w_m)(t,z) \quad = \quad -\infty. \tag{1.5.15}$$

Proof. We set

$$f_1(t,z) \;=\; e^{\rho'(T-t)} \tilde{L}(z)^{\gamma'}, \qquad f_2(t,z) \;=\; e^{\rho'(T-t)} \left(\frac{e^{\lambda y}}{p} + p e^{-\lambda y} \right).$$

From (1.4.4), we have for all $t \in [0,T)$, $z \in \mathcal{S} \setminus \tilde{\mathcal{N}} = \{ z \in \mathcal{S} \; : \tilde{L}(z) \geq k \}$:

$$f_1(t, \Gamma(z,\zeta)) \quad \leq \quad e^{\rho'(T-t)} (\tilde{L}(z) - k)^{\gamma'}, \quad \forall \zeta \in \mathcal{C}(z),$$

and so

$$(f_1 - \mathcal{H} f_1)(t,z) \quad \geq \quad e^{\rho'(T-t)} \left[\tilde{L}(z)^{\gamma'} - (\tilde{L}(z) - k)^{\gamma'} \right] \; > \; 0 \tag{1.5.16}$$

Notice that relation (1.5.16) holds trivially true when $z \in \mathcal{N}$ since in this case $\mathcal{C}(z) = \emptyset$ and so $\mathcal{H}f(t,z) = -\infty$. We then deduce that for any compact set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$, there exists some constant $\delta_0 > 0$ s.t.

$$f_1 - \mathcal{H}f_1 \;\; \geq \;\; \delta_0, \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K}.$$

Moreover, a direct calculation shows that for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$, $\zeta \in \mathcal{C}(z)$, $f_2(t, \Gamma(z,\zeta))$ $= f_2(t,z)$, and so

$$f_2 - \mathcal{H}f_2 \;\; = \;\; 0.$$

This implies

$$\begin{aligned} \phi_\nu - \mathcal{H}\phi_\nu \;\; &= \;\; f_1 + \nu f_2 \; - \; \mathcal{H}(f_1 + \nu f_2) \; \geq \; (f_1 - \mathcal{H}f_1) + \nu(f_2 - \mathcal{H}f_2) \\ &\geq \;\; \delta_0, \qquad \text{on} \;\; [0,T) \times \mathcal{S} \cap \mathcal{K}. \end{aligned} \tag{1.5.17}$$

On the other hand, the same calculation as in (1.4.23) shows that for ρ' large enough, actually strictly larger than $\frac{\gamma'}{1-\gamma'} \frac{b^2 + r^2 + \sigma^2 r(1-\gamma')}{\sigma^2}$, we have $-\dfrac{\partial f_1}{\partial t} - \mathcal{L}f_1 > 0$ on $[0,T) \times \mathcal{S}$. Hence, for any compact set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$, there exists some constant $\delta_1 > 0$ s.t.

$$-\frac{\partial f_1}{\partial t} - \mathcal{L}f_1 \;\; \geq \;\; \delta_1 \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K}.$$

A direct calculation also shows that for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$:

$$-\frac{\partial f_2}{\partial t}(t,z) - \mathcal{L}f_2(t,z) \;\; = \;\; e^{\rho'(T-t)} \left[(\rho' + b - \sigma^2) \frac{e^{\lambda y}}{p} + (\rho' - b) p e^{-\lambda y} \right] \; \geq \; 0,$$

since $\rho' \geq (\sigma^2 - b) \vee b$. This implies that for any compact set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$, there exists some constant $\delta_1 > 0$ s.t.

$$\begin{aligned} -\frac{\partial \phi_\nu}{\partial t} - \mathcal{L}\phi_\nu \;\; &= \;\; -\frac{\partial f_1}{\partial t} - \mathcal{L}f_1 + \nu \left(-\frac{\partial f_2}{\partial t} - \mathcal{L}f_2 \right) \\ &\geq \;\; \delta_1 \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K}. \end{aligned} \tag{1.5.18}$$

By writing the viscosity supersolution property of w, we deduce from the inequalities (1.5.17)-(1.5.18) the viscosity supersolution of w_m to

$$\min \left[-\frac{\partial w_m}{\partial t} - \mathcal{L}w_m, w_m - \mathcal{H}w_m \right] \;\; \geq \;\; \frac{\delta}{m} \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K},$$

and so (1.5.14), where we set $\delta = \delta_0 \wedge \delta_1$. Finally, since $u, w \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$, we have for some positive constant K :

$$\begin{aligned} (u - w_m)(t,z) \;\; &\leq \;\; K \left[1 + \left(x + \frac{p}{\lambda} \right)^\gamma \right] - \frac{1}{m} \left[\left(x + \frac{p}{\lambda} \right)^{\gamma'} + \nu \left(\frac{e^{\lambda y}}{p} + p e^{-\lambda y} \right) \right] \\ &\longrightarrow \;\; -\infty, \quad \text{as} \;\; |z| \to \infty, \end{aligned}$$

since $\gamma' > \gamma$ and $\nu > 0$. □

We now follow general viscosity solution technique, based on the Ishii technique (see [19]) and adapt arguments from [40], [55] for handling with specificities coming from the nonlocal intervention operator $\mathcal{H}$ and [5], [1] for the boundary conditions. The general idea is to build a test function so that the minimum associated with the (strict) supersolution cannot be on the boundary. However, the usual method in [62] does not apply here since it requires the continuity of the supersolution on the boundary, which is precisely not the case in our model. Instead, we adapt a method in [5], which requires the smoothness of the boundary. This is the case here except on the part D_0 of the boundary, but for which one has proved directly in Corollary 1.4.6 the continuity of the value function.

Proof of Theorem 1.5.2
Let u and w as in Theorem 1.5.2. We (re)define w on $[0,T) \times \partial\mathcal{S}$ by :

$$w(t,z) \quad = \quad \liminf_{\substack{(t',z') \to (t,z) \\ (t',z') \in [0,T) \times \mathcal{S}}} \quad w(t',z'), \quad \forall (t,z) \in [0,T) \times \partial\mathcal{S}, \tag{1.5.19}$$

and construct a strict viscosity supersolution to (1.5.1) according to Lemma 1.5.2, by considering for $m \geq 1$, the usc function on $[0,T) \times \bar{\mathcal{S}}$:

$$w_m \quad = \quad w + \frac{1}{m}\phi_\nu, \tag{1.5.20}$$

where ϕ_ν is given in (1.5.13) for some given $\nu > 0$. Recalling the definitions (1.5.12) of u and w on $\{T\} \times \bar{\mathcal{S}}$, we have then an extension of u and w_m, which are usc and lsc on $[0,T] \times \bar{\mathcal{S}}$. In order to prove the comparison principle, it is sufficient to show that $\sup_{[0,T]\times\bar{\mathcal{S}}}(u - w_m)$ ≤ 0 for all $m \geq 1$, since the required result is obtained by letting m to infinity. We argue by contradiction and suppose that there exists some $m \geq 1$ s.t.

$$\mu \ := \ \sup_{[0,T]\times\bar{\mathcal{S}}} (u - w_m) \quad > \quad 0.$$

Since $u - w_m$ is usc on $[0,T] \times \bar{\mathcal{S}}$, $\lim_{|z|\to\infty}(u - w_m)(z) = -\infty$ by (1.5.15), $(u - w_m)(T,.)$ ≤ 0 by (1.5.12), and $(u - w_m)(t,z) \leq 0$ for $(t,z) \in [0,T) \times D_0$ by (1.5.11), there exists a an open set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$ with closure $\bar{\mathcal{K}}$ compact s.t.

$$\operatorname{Arg}\max_{[0,T]\times\bar{\mathcal{S}}}(u - w_m) \ \neq \ \emptyset \quad \subset \quad [0,T) \times \bar{\mathcal{S}} \setminus D_0 \cap \mathcal{K}.$$

Take then some $(t_0, z_0) \in [0,T) \times \bar{\mathcal{S}} \setminus D_0 \cap \mathcal{K}$ s.t. $\mu = (u - w_m)(t_0, z_0)$ and distinguish the two cases :

• <u>*Case 1.* : $z_0 \in \partial\mathcal{S} \setminus D_0 \cap \mathcal{K}$.</u>
⋆ From (1.5.19), there exists a sequence $(t_i, z_i)_{i\geq 1}$ in $[0,T) \times \mathcal{S} \cap \mathcal{K}$ converging to (t_0, z_0) s.t. $w_m(t_i, z_i)$ tends to $w_m(t_0, z_0)$ when i goes to infinity. We then set $\beta_i = |t_i - t_0|$, $\varepsilon_i =$

$|z_i - z_0|$ and consider the function Φ_i defined on $[0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$ by :

$$\begin{aligned}
\Phi_i(t,t',z,z') &= u(t,z) - w_m(t',z') - \varphi_i(t,t',z,z') \qquad (1.5.21)\\
\varphi_i(t,t',z,z') &= |t-t_0|^2 + |z-z_0|^4 + \frac{|t-t'|^2}{2\beta_i} + \frac{|z-z'|^2}{2\varepsilon_i} + \left(\frac{d(z')}{d(z_i)} - 1\right)^4.
\end{aligned}$$

Here $d(z)$ denotes the distance from z to $\partial\mathcal{S}$. We claim that for $z_0 \notin D_0$, there exists an open neighborhood $\mathcal{V}_0 \subset \mathcal{K}$ of z_0 in which this distance function $d(.)$ is twice continuously differentiable with bounded derivatives. This is well-known (see e.g. [31]) when z_0 lies in the smooth parts $\partial\mathcal{S} \setminus (D_k \cup C_1 \cup C_2)$ of the boundary $\partial\mathcal{S}$. This is also true for $z_0 \in D_k \cup C_1 \cup C_2$. Indeed, at these corner lines of the boundary, the inner normal vectors form an acute angle (positive scalar product) and thus one can extend from z_0 the boundary to a smooth boundary so that the distance d is equal, locally on a neighborhood of z_0, to the distance to this smooth boundary. Notice that this is not true when $z_0 \in D_0$, which forms a right angle. Now, since Φ_i is usc on the compact set $[0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$, there exists $(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) \in [0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$ that attains its maximum on $[0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$:

$$\mu_i \;:=\; \sup_{[0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2} \Phi_i(t,t',z,z') \;=\; \Phi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i).$$

Moreover, there exists a subsequence, still denoted $(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i)_{i\geq 1}$, converging to some $(\hat{t}_0, \hat{t}'_0, \hat{z}_0, \hat{z}'_0) \in [0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$. By writing that $\Phi_i(t_0, t_i, z_0, z_i) \leq \Phi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i)$, we have :

$$\begin{aligned}
& u(t_0,z_0) - w_m(t_i,z_i) - \frac{1}{2}\left(|t_i - t_0| + |z_i - z_0|\right) && (1.5.22)\\
\leq\ & \mu_i \;=\; u(\hat{t}_i, \hat{z}_i) - w_m(\hat{t}'_i, \hat{z}'_i) - \left(|\hat{t}_i - t_0|^2 + |\hat{z}_i - z_0|^4\right) - R_i && (1.5.23)\\
\leq\ & u(\hat{t}_i, \hat{z}_i) - w_m(\hat{t}'_i, \hat{z}'_i) - \left(|\hat{t}_i - t_0|^2 + |\hat{z}_i - z_0|^4\right), && (1.5.24)
\end{aligned}$$

where we set

$$R_i \;=\; \frac{|\hat{t}_i - \hat{t}'_i|^2}{2\beta_i} + \frac{|\hat{z}_i - \hat{z}'_i|^2}{2\varepsilon_i} + \left(\frac{d(\hat{z}'_i)}{d(z_i)} - 1\right)^4.$$

From the boundedness of u, w_m on $[0,T]\times\bar{\mathcal{S}}\cap\bar{\mathcal{K}}$, we deduce by inequality (1.5.23) the boundedness of the sequence $(R_i)_{i\geq 1}$, which implies $\hat{t}_0 = \hat{t}'_0$ and $\hat{z}_0 = \hat{z}'_0$. Then, by sending i to infinity into (1.5.22) and (1.5.24), with the upper-semicontinuity (resp. lower-semicontinuity) of u (resp. w_m), we obtain $\mu = u(t_0,z_0) - w_m(t_0,z_0) \leq u(\hat{t}_0,\hat{z}_0) - w_m(\hat{t}_0,\hat{z}_0) - |\hat{t}_0 - t_0|^2 - |\hat{z}_0 - z_0|^4$. By the definition of μ, this shows :

$$\hat{t}_0 \;= \hat{t}'_0 \;=\; t_0, \qquad \hat{z}_0 \;= \hat{z}'_0 \;=\; z_0. \qquad (1.5.25)$$

Sending again i to infinity into (1.5.22)-(1.5.23)-(1.5.24), we thus derive that $\mu \leq \lim_i \mu_i = \mu - \lim_i R_i \leq \mu$, and so

$$\begin{aligned}
\mu_i &\;\longrightarrow\; \mu && (1.5.26)\\
\frac{|\hat{t}_i - \hat{t}'_i|^2}{2\beta_i} + \frac{|\hat{z}_i - \hat{z}'_i|^2}{2\varepsilon_i} + \left(\frac{d(\hat{z}'_i)}{d(z_i)} - 1\right)^4 &\;\rightarrow\; 0, && (1.5.27)
\end{aligned}$$

as i goes to infinity. In particular, for i large enough , we have $\hat{t}_i$, $\hat{t}'_i < T$ (since $t_0 < T$), $d(\hat{z}'_i) \geq d(z_i)/2 > 0$, and so $\hat{z}'_i \in \mathcal{S}$. For i large enough, we may also assume that $\hat{z}_i$, $\hat{z}'_i$ lie in the neighborhood $\mathcal{V}_0$ of z_0 so that the derivatives upon order 2 of d at $\hat{z}_i$ and $\hat{z}'_i$ exist and are bounded.

$\star$ We may then apply Ishii's lemma (see Theorem 8.3 in [19]) to $(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) \in [0,T) \times [0,T) \times \bar{\mathcal{S}} \cap \mathcal{V}_0 \times \mathcal{S} \cap \mathcal{V}_0$ that attains the maximum of Φ_i in (1.5.21). Hence, there exist 3×3 matrices $M = (M_{jl})_{1 \leq j,l \leq 3}$ and $M' = (M'_{jl})_{1 \leq j,l \leq 3}$ s.t. :

$$\begin{aligned} (q_0, q, M) &\in \bar{J}^{2,+} u(\hat{t}_i, \hat{z}_i), \\ \left(q'_0, q', M'\right) &\in \bar{J}^{2,-} w_m(\hat{t}'_i, \hat{z}'_i) \end{aligned}$$

where

$$q_0 = \frac{\partial \varphi_i}{\partial t}(\hat{t}_i, \hat{t}'_i, \hat{z}, \hat{t}'_i), \qquad q = (q_j)_{1 \leq j \leq 3} = D_z \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) \tag{1.5.28}$$

$$q'_0 = -\frac{\partial \varphi_i}{\partial t}(\hat{t}_i, \hat{t}'_i, \hat{z}, \hat{t}'_i), \qquad q' = (q'_j)_{1 \leq j \leq 3} = -D_{z'} \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i). \tag{1.5.29}$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -M' \end{pmatrix} \leq D^2_{z,z'} \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) + \varepsilon_i \left(D^2_{z,z'} \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i)\right)^2 \tag{1.5.30}$$

By writing the viscosity subsolution property (1.5.4) of u and the strict viscosity supersolution (1.5.14) of w_m, we have :

$$\min\left[-q_0 - r\hat{x}_i q_1 - b\hat{p}_i q_3 - \frac{1}{2}\sigma^2 \hat{p}_i^2 M_{33}, u(\hat{t}_i, \hat{z}_i) - \mathcal{H}u(\hat{t}_i, \hat{z}_i)\right] \leq 0 \tag{1.5.31}$$

$$\min\left[-q'_0 - r\hat{x}'_i q'_1 - b\hat{p}'_i q'_3 - \frac{1}{2}\sigma^2 \hat{p}_i'^2 M'_{33}, w_m(\hat{t}'_i, \hat{z}'_i) - \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i)\right] \geq \frac{\delta}{m}. \tag{1.5.32}$$

We then distinguish the following two possibilities in (1.5.31) :

1. $u(\hat{t}_i, \hat{z}_i) - \mathcal{H}u(\hat{t}_i, \hat{z}_i) \leq 0$.

Since, from (1.5.32), we also have: $w_m(\hat{t}'_i, \hat{z}'_i) - \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i) \geq \frac{\delta}{m}$, we obtain by combining these two inequalities :

$$\mu_i \leq u(\hat{t}_i, \hat{z}_i) - w_m(\hat{t}'_i, \hat{z}'_i) \leq \mathcal{H}u(\hat{t}_i, \hat{z}_i) - \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i) - \frac{\delta}{m}$$

Sending i to ∞, and by (1.5.26), we obtain :

$$\begin{aligned} \mu &\leq \limsup_{i \to \infty} \mathcal{H}u(\hat{t}_i, \hat{z}_i) - \liminf_{i \to \infty} \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i) - \frac{\delta}{m} \\ &\leq \mathcal{H}u(t_0, z_0) - \mathcal{H}w_m(t_0, z_0) - \frac{\delta}{m}, \end{aligned}$$

from (1.5.25) and where we used the upper-semicontinuity of $\mathcal{H}u$ and the lower-semicontinuity of $\mathcal{H}w_m$ (see Lemma 1.5.1). By compactness of $\mathcal{C}(z_0)$, and since u is usc, there exists some

$\zeta_0 \in \mathcal{C}(z_0)$ s.t. $\mathcal{H}u(t_0, z_0) = u(t_0, \Gamma(z_0, \zeta_0))$. We then get the desired contradiction :

$$\begin{aligned}\mu &\leq \mathcal{H}u(t_0, z_0) - \mathcal{H}w_m(t_0, z_0) - \frac{\delta}{m} \\ &\leq u(t_0, \Gamma(z_0, \zeta_0)) - w_m(t_0, \Gamma(z_0, \zeta_0)) - \frac{\delta}{m} \leq \mu - \frac{\delta}{m}.\end{aligned}$$

2. $-q_0 - r\hat{x}_i q_1 - b\hat{p}_i q_3 - \frac{1}{2}\sigma^2 \hat{p}_i^2 M_{33} \leq 0$.
Since, from (1.5.32), we also have: $-q_0' - r\hat{x}_i' q_1' - b\hat{p}_i' q_3' - \frac{1}{2}\sigma^2 \hat{p}_i'^2 M_{33}' \geq \frac{\delta}{m}$, we obtain by combining these two inequalities :

$$-(q_0 - q_0') - r(\hat{x}_i q_1 - \hat{x}_i' q_1') - b(\hat{p}_i q_3 - \hat{p}_i' q_3') - \frac{1}{2}\sigma^2(\hat{p}_i^2 M_{33} - \hat{p}_i'^2 M_{33}') \leq -\frac{\delta}{m}. \quad (1.5.33)$$

Now, from (1.5.28)-(1.5.29), we explicit :

$$\begin{aligned} q_0 &= 2(\hat{t}_i - t_0) + \frac{(\hat{t}_i - \hat{t}_i')}{\beta_i}, & q &= 4(\hat{z}_i - z_0)|\hat{z}_i - z_0|^2 + \frac{(\hat{z}_i - \hat{z}_i')}{\varepsilon_i} \\ q_0' &= \frac{(\hat{t}_i - \hat{t}_i')}{\beta_i}, & q' &= \frac{(\hat{z}_i - \hat{z}_i')}{\varepsilon_i} - 4Dd(\hat{z}_i')\left(\frac{d(\hat{z}_i')}{d(z_i)} - 1\right)^3 \end{aligned}$$

and we see by (1.5.25) and (1.5.27) that $q_0 - q_0'$, $\hat{x}_i q_1 - \hat{x}_i' q_1'$ and $\hat{p}_i q_3 - \hat{p}_i' q_3'$ converge to zero when i goes to infinity. Moreover, from (1.5.30), we have :

$$\frac{1}{2}\sigma^2 \hat{p}_i^2 M_{33} - \frac{1}{2}\sigma^2 \hat{p}_i'^2 M_{33}' \leq \mathcal{E}_i, \quad (1.5.34)$$

where

$$\begin{aligned}\mathcal{E}_i &= A_i\left(D^2_{z,z'}\varphi_i(\hat{t}_i, \hat{t}_i', \hat{z}_i, \hat{z}_i') + \varepsilon_i\left(D^2_{z,z'}\varphi_i(\hat{t}_i, \hat{t}_i', \hat{z}_i, \hat{z}_i')\right)^2\right) A_i^{\intercal} \\ &= A_i\left(\begin{pmatrix} \frac{1}{\varepsilon_i}I_3 + P_i & -\frac{1}{\varepsilon_i}I_3 \\ -\frac{1}{\varepsilon_i}I_3 & \frac{1}{\varepsilon_i}I_3 + Q_i \end{pmatrix} + \varepsilon_i \begin{pmatrix} \frac{1}{\varepsilon_i}I_3 + P_i & -\frac{1}{\varepsilon_i}I_3 \\ -\frac{1}{\varepsilon_i}I_3 & \frac{1}{\varepsilon_i}I_3 + Q_i \end{pmatrix}^2\right) A_i^{\intercal}\end{aligned}$$

with

$$\begin{aligned} A_i &= \left(0, 0, \hat{p}_i, 0, 0, \hat{p}_i'\right), \quad P_i = 4|\hat{z}_i - z_0|^2 I_3 + 8(\hat{z}_i - z_0)(\hat{z}_i - z_0)^{\intercal} \\ Q_i &= 12\left(\frac{d(\hat{z}_i')}{d(\hat{z}_i')} - 1\right)^2 Dd(\hat{z}_i')Dd(\hat{z}_i')^{\intercal} + 4\left(\frac{d(\hat{z}_i')}{d(\hat{z}_i')} - 1\right)^3 D^2 d(\hat{z}_i'). \end{aligned}$$

Here $^{\intercal}$ denotes the transpose operator. After some straightforward calculation, we then get :

$$\mathcal{E}_i = 3\frac{(\hat{p}_i' - \hat{p}_i)^2}{\varepsilon_i} + A_i\left(\begin{pmatrix} 3P_i & -2Q_i \\ -2P_i & 3Q_i \end{pmatrix} + \varepsilon_i \begin{pmatrix} P_i^2 & 0 \\ 0 & Q_i^2 \end{pmatrix}\right) A_i^{\intercal},$$

which converges also to zero from (1.5.25) and (1.5.27). Therefore, by sending i to infinity into (1.5.33), we see that the lim sup of its l.h.s. is nonnegative, which gives the required contradiction : $0 \leq -\delta/m$.

• *Case 2. :* $z_0 \in \mathcal{S} \cap \mathcal{K}$.
This case is dealt similarly as in Case 1. and its proof is omitted. It suffices e.g. to consider the function

$$\begin{aligned} \Psi_i(t,z,z') &= u(t,z) - w_m(t,z') - \psi_i(t,z,z') \\ \psi_i(t,z,z') &= |t-t_0|^2 + |z-z_0|^4 + \frac{i}{2}|z-z'|^2, \end{aligned}$$

for $i \geq 1$, and to take a maximum $(\tilde{t}_i, \tilde{z}_i, \tilde{z}'_i)$ of Ψ_i. We then show that the sequence $(\tilde{t}_i, \tilde{z}_i, \tilde{z}'_i)_{i\geq 1}$ converges to (t_0, z_0, z_0) as i goes to infinity and we apply Ishii's lemma to get the required contradiction.

By combining previous results, we then finally obtain the following PDE characterization of the value function.

Corollary 1.5.1 *The value function v is continuous on $[0,T) \times \mathcal{S}$ and is the unique (in $[0,T) \times \mathcal{S}$) constrained viscosity solution to* (1.5.1) *lying in $\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$ and satisfying the boundary condition :*

$$\lim_{(t',z')\to(t,z)} v(t',z') = 0, \quad \forall (t,z) \in [0,T) \times D_0,$$

and the terminal condition

$$v(T^-,z) := \lim_{\substack{(t,z')\to(T,z)\\ t<T,\, z'\in\mathcal{S}}} v(t,z') = \bar{U}(z), \quad \forall z \in \bar{\mathcal{S}}.$$

Proof. From Theorem 1.5.1, v^* is an usc viscosity subsolution to (1.5.1) in $[0,T) \times \bar{\mathcal{S}}$ and v_* is a lsc viscosity supersolution to (1.5.1) in $[0,T) \times \mathcal{S}$. Moreover, by Corollary 1.4.6 and Proposition 1.4.3, we have $v^*(t,z) = v_*(t,z) = 0$ for all $(t,z) \in [0,T) \times D_0$, and $v^*(T,z) = v_*(T,z) = \bar{U}(z)$ for all $z \in \bar{\mathcal{S}}$. Then by Theorem 1.5.2, we deduce $v^* \leq v_*$ on $[0,T] \times \mathcal{S}$, which proves the continuity of v on $[0,T) \times \mathcal{S}$. On the other hand, suppose that $\tilde{v}$ is another constrained viscosity solution to (1.5.1) with $\lim_{(t',z')\to(t,z)} v(t',z') = 0$ for $(t,z) \in [0,T) \times D_0$ and $\tilde{v}(T^-,z) = \bar{U}(z)$ for $z \in \bar{\mathcal{S}}$. Then, $\tilde{v}^*(t,z) = v_*(t,z) = v^*(t,z) = \tilde{v}_*(t,z)$ for $(t,z) \in [0,T) \times D_0$ and $\tilde{v}^*(T,z) = v_*(T,z) = v^*(T,z) = \tilde{v}_*(T,z)$ for z in $\bar{\mathcal{S}}$. We then deduce by Theorem 1.5.2 that $v^* \leq \tilde{v}_* \leq \tilde{v}^* \leq v_*$ on $[0,T] \times \mathcal{S}$. This proves $v = \tilde{v}$ on $[0,T) \times \mathcal{S}$. □

1.6 Conclusion

We formulated a model for optimal portfolio selection under liquidity risk and price impact. Our main result is a characterization of the value function as the unique constrained viscosity solution to the quasi-variational Hamilton-Jacobi-Bellman inequality associated to this impulse control problem under solvency constraint. The main technical difficulties come from the nonlinearity due to price impact, and the state constraint. They are overcome with the specific exponential form of the price impact function : a natural theoretical question is to extend our results for general price impact functions. Once we have provided a complete PDE characterization of the value function, the next step, from an applied view point, is to numerically solve this quasi-variational inequality, see Chapter 2. Moreover, from an economic viewpoint, it would be of course interesting to analyse the effects of liquidity risk and price impact in our model on the optimal portfolio in a classical market without frictions, e.g. the Merton model.

Appendix : Proof of Lemma 1.4.1

We first prove the following elementary lemma.

Lemma 1.A.1 *For any $y \in \mathbb{R}$, there exists an unique $\bar{\zeta}(y) \in \mathbb{R}$ s.t.*

$$\bar{g}(y) \ := \ \max_{\zeta \in \mathbb{R}} g(y,\zeta) \ = \ \bar{\zeta}(y)(e^{-\lambda y} - e^{\lambda \bar{\zeta}(y)}). \tag{1.A.1}$$

The function $\bar{g}$ is differentiable, decreasing on $(-\infty, 0)$, increasing on $(0,\infty)$, with $\bar{g}(0) = 0$, $\lim_{y\to-\infty} \bar{g}(y) = \infty$, $\lim_{y\to\infty} \bar{g}(y) = e^{-1}/\lambda$, and for all $p > 0$,

$$\ell(y,p) + p\bar{g}(y) < 0 \ \text{ if } \ y < 0 \quad \text{ and } \quad -\ell(y,p) + p\bar{g}(y) < 0 \ \text{ if } \ 0 < y \le \frac{1}{\lambda}.$$

Proof. (i) For fixed y, a straightforward study of the differentiable function $\zeta \to g_y(\zeta) := g(y,\zeta)$ shows that there exists an unique $\bar{\zeta}(y) \in \mathbb{R}$ such that :

$$\begin{aligned} G(y,\bar{\zeta}(y)) \ = \ g'_y(\bar{\zeta}(y)) \quad &= \quad e^{-\lambda y} - e^{\lambda\bar{\zeta}(y)}(1+\lambda\bar{\zeta}(y)) \ = \ 0, \\ g'_y(\zeta) > (\text{ resp. } <)\ 0 \quad &\Longleftrightarrow \quad \zeta < (\text{ resp. } >)\ \bar{\zeta}(y) \end{aligned}$$

This proves that g_y is increasing on $(-\infty, \bar{\zeta}(y))$ and decreasing on $(\bar{\zeta}(y), \infty)$ with

$$\max_{\zeta\in\mathbb{R}} g_y(\zeta) \ = \ g_y(\bar{\zeta}(y)) \ := \ \bar{g}(y),$$

i.e. (1.A.1). Since $g'_y(-1/\lambda) = e^{-\lambda y} > 0$, we notice that $\bar{\zeta}(y)$ is valued in $(-1/\lambda, \infty)$ for all $y \in \mathbb{R}$. Moreover, since the differentiable function $(y,\zeta) \to G(y,\zeta) := g'_y(\zeta)$ is decreasing in y on $\mathbb{R}$: $\dfrac{\partial G}{\partial y} < 0$ and decreasing in ζ on $(-1/\lambda, \infty)$: $\dfrac{\partial G}{\partial \zeta} < 0$, we derive by the implicit functions theorem that $\bar{\zeta}(y)$ is a differentiable decreasing function on $\mathbb{R}$. Since $G(y, -1/\lambda)$ $= e^{-\lambda y}$ goes to zero as y goes to infinity, we also obtain that $\bar{\zeta}(y)$ goes to $-1/\lambda$ as y goes to infinity. By noting that for all ζ, $G(y,\zeta)$ goes to ∞ when y goes to $-\infty$, we deduce that $\bar{\zeta}(y)$ goes to ∞ as y goes to $-\infty$. Since $G(0,0) = 0$, we also have $\bar{\zeta}(0) = 0$. Notice also that $G(y,-y) = \lambda y e^{-\lambda y}$: hence, when $y < 0$, $G(y,-y) < 0 = G(y,\bar{\zeta}(y))$ so that $\bar{\zeta}(y) < -y$, and when $y > 0$, $G(y,-y) > 0 = G(y,\bar{\zeta}(y))$ so that $\bar{\zeta}(y) > -y$.

(ii) By the envelope theorem, the function $\bar{g}$ defined by $\bar{g}(y) = \max_{\zeta\in\mathbb{R}} g(y,\zeta) = g(y,\bar{\zeta}(y))$ is differentiable on $\mathbb{R}$ with

$$\bar{g}'(y) \;=\; \frac{\partial g}{\partial y}(y,\bar{\zeta}(y)) \;=\; -\lambda\bar{\zeta}(y)e^{-\lambda\bar{\zeta}(y)}, \quad y \in \mathbb{R}.$$

Since $\bar{\zeta}(y) >$ (resp. $<$) 0 iff $y <$ (resp. $>$) 0 with $\bar{\zeta}(0) = 0$, we deduce the decreasing (resp. increasing) property of $\bar{g}$ on $(-\infty, 0)$ (resp. $(0,\infty)$) with $\bar{g}(0) = 0$. Since $\bar{\zeta}(y)$ converges to $-1/\lambda$ as y goes to infinity, we immediately see from expression (1.A.1) of $\bar{g}$ that $\bar{g}(y)$ converges to e^{-1}/λ as y goes to infinity. For $y < 0$ and by taking $\zeta = -y/2$ in the maximum in (1.A.1), we have $\bar{g}(y) \geq -y(e^{-\lambda y} - e^{-\lambda y/2})/2$, which shows that $\bar{g}(y)$ goes to infinity as y goes to $-\infty$. When $y < 0$, we have $0 < \bar{\zeta}(y) < -y$, and thus by (1.A.1), we get :

$$\bar{g}(y) \;<\; -y\left(e^{-\lambda y} - e^{\lambda\bar{\zeta}(y)}\right),$$

and so $\ell(y,p) + p\bar{g}(y) < pye^{\lambda\bar{\zeta}(y)} < 0$ for all $p > 0$. When $y > 0$, we have $\bar{\zeta}(y) < 0$ and thus by (1.A.1), we get : $\bar{g}(y) < -\bar{\zeta}(y)e^{\lambda\bar{\zeta}(y)}$. Now, since the function $\zeta \mapsto -\zeta e^{\lambda\zeta}$ is decreasing on $[-1/\lambda, 0]$, we have for all $0 < y \leq 1/\lambda$, $-1/\lambda \leq -y < \bar{\zeta}(y)$ and so

$$\bar{g}(y) \;<\; ye^{-\lambda y}.$$

This proves $p\bar{g}(y) \leq \ell(y,p)$ for all $0 < y \leq 1/\lambda$ and $p > 0$. □

Proof of Lemma 1.4.1. For any $z \in \bar{\mathcal{S}}$, we write $\mathcal{C}(z) = \mathcal{C}_0(z) \cup \mathcal{C}_1(z)$ where $\mathcal{C}_0(z) = \{\zeta \in \mathbb{R} \;: L_0(\Gamma(z,\zeta)) \geq 0\}$ and $\mathcal{C}_1(z) = \{\zeta \in \mathbb{R} \;: L_1(\Gamma(z,\zeta)) \geq 0, y + \zeta \geq 0\}$. From (1.4.1) and by noting that the function $\zeta \mapsto pg(y,\zeta)$ goes to $-\infty$ as $|\zeta|$ goes to infinity, we see that $\mathcal{C}_0(z)$ is bounded. Since the function $\zeta \mapsto p\theta(\zeta,p)$ goes to infinity as ζ goes to infinity, we also see that $\mathcal{C}_1(z)$ is bounded. Hence, $\mathcal{C}(z)$ is bounded. Moreover, for any $z = (x,y,p) \in \bar{\mathcal{S}}$, the function $\zeta \mapsto L(\Gamma(z,\zeta))$ is upper-semicontinuous : it is indeed continuous on $\mathbb{R}\setminus\{-y\}$ and upper-semicontinuous on $-y$. This implies the closure property and then the compactness of $\mathcal{C}(z)$.

$\star$ Fix some arbitrary $z \in \partial^y\mathcal{S}$. Then, for any $\zeta \in \mathbb{R}$, we have $L_0(\Gamma(z,\zeta)) = x - k + pg(0,\zeta) - k$. Since $g(0,\zeta) \leq 0$ for all $\zeta \in \mathbb{R}$ and $x \leq k$, we see that $L(\Gamma(z,\zeta)) < 0$ for all $\zeta \in \mathbb{R}$. On the other hand, we have $L_1(\Gamma(z,\zeta)) = x - \theta(\zeta,p) - k$. Since $\theta(\zeta,p) \geq 0$ for all $\zeta \geq 0$, and recalling that $x < k$, we also see that $L_1(\Gamma(z,\zeta)) = x - \theta(\zeta,p) - k < 0$ for all $\zeta \geq 0$. Therefore $L(\Gamma(z,\zeta)) < 0$ for all $\zeta \in \mathbb{R}$ and so $\mathcal{C}(z)$ is empty.

$\star$ Fix some arbitrary $z \in \partial_0^x\mathcal{S}$. Then, for any $\zeta \in \mathbb{R}$, we have $L_0(\Gamma(z,\zeta)) = \ell(y,p) - k + pg(y,\zeta) - k$. Now, we recall from Remark 1.2.2 that $\ell(y,p) \leq p/(\lambda e) < k$. Moreover, by Lemma 1.A.1, we have $pg(y,\zeta) \leq p\bar{g}(y) \leq p/(\lambda e) < k$. This implies $L_0(\Gamma(z,\zeta)) < 0$ for any $\zeta \in \mathbb{R}$. On the other hand, we have $L_1(\Gamma(z,\zeta)) = -\theta(\zeta,p) - k$. Since $\theta(\zeta,p) \geq -p/(\lambda e)$ for all $\zeta \in \mathbb{R}$, we get $L_1(\Gamma(z,\zeta)) \leq p/(\lambda e) - k < 0$. Therefore $\mathcal{C}(z)$ is empty.

$\star$ Fix some arbitrary $z \in \partial_1^x\mathcal{S}$. Then, for any $\zeta \in \mathbb{R}$, we have $L_0(\Gamma(z,\zeta)) = \ell(y,p) - k + pg(y,\zeta) - k$. Now, we recall from Remark 1.2.2 that $\ell(y,p) < k$. Moreover, since $0 < y \leq$

$1/\lambda$, we get from Lemma 1.A.1 : $pg(y,\zeta) \le p\bar{g}(y) < \ell(y,p) < k$ for all $\zeta \in \mathbb{R}$. This implies $L_0(\Gamma(z,\zeta)) < 0$ for any $\zeta \in \mathbb{R}$. On the other hand, we have $L_1(\Gamma(z,\zeta)) = -\theta(\zeta,p) - k$. Since the function $\zeta \mapsto \theta(\zeta,p)$ is increasing on $[-1/\lambda,\infty)$ and $y < 1/\lambda$, we have for all $\zeta \ge -y$, $\theta(\zeta,p) \ge \theta(-y,p) = -\ell(y,p)$, and so $-\theta(\zeta,p) - k \le \ell(y,p) - k < 0$. This implies $L_1(\Gamma(z,\zeta)) < 0$ for all $\zeta \in \mathbb{R}$ and thus $\mathcal{C}(z)$ is empty.

$\star$ Fix some arbitrary $z \in \partial_2^x\mathcal{S}$. Then for $\zeta = -1/\lambda$, we have $\theta(\zeta,p) = -p/(\lambda e)$ and $y + \zeta > 0$ (see Remark 1.2.2). Hence, $L(\Gamma(z,-1/\lambda)) \ge L_1(\Gamma(z,-1/\lambda)) \ge 0$ and so $-1/\lambda \in \mathcal{C}(z)$. Moreover, take some arbitrary $\zeta \in \mathcal{C}(z) = \mathcal{C}_0(z) \cup \mathcal{C}_1(z)$. In the case where $\zeta \in \mathcal{C}_0(z)$, i.e. $L_0(\Gamma(z,\zeta)) = \ell(y,p) - k + pg(y,\zeta) - k \ge 0$, and recalling that $\ell(y,p) < k$, we must have necessarily $g(y,\zeta) > 0$. This implies $-y < \zeta < 0$. Similarly, when $\zeta \in \mathcal{C}_1(z)$, i.e. $-\theta(\zeta,p)-k \ge 0$ and $y+\zeta \ge 0$, we must have $-y < \zeta < 0$. Therefore, $\mathcal{C}(z) \subset (-y,0)$.

$\star$ Fix some arbitrary $z \in \partial_\ell^-\mathcal{S} \cup \partial_\ell^+\mathcal{S}$. Then we have $L(\Gamma(z,-y)) = L_1(\Gamma(z,-y)) = 0$, which shows that $\zeta = -y \in \mathcal{C}(z)$. Consider now the case where $z \in \partial_\ell^-\mathcal{S} \cup \partial_\ell^{+,\lambda}\mathcal{S}$. We claim that $\mathcal{C}_1(z) = \{-y\}$. Indeed, take some $\zeta \in \mathcal{C}_1(z)$, i.e. $x - \theta(\zeta,p) - k \ge 0$ and $y + \zeta \ge 0$. Then, $\theta(\zeta,p) \ge \theta(-y,p) = -\ell(y,p)$ (since $\zeta \mapsto \theta(\zeta,p)$ is increasing on $[-1/\lambda,\infty)$ and $-y \ge -1/\lambda$) and so $0 \le x - \theta(\zeta,p) - k \le x + \ell(y,p) - k = 0$. Hence, we must have $\zeta = -y$. Take now some arbitrary $\zeta \in \mathcal{C}_0(z)$. Hence, $L_0(\Gamma(z,\zeta)) = pg(y,\zeta) - k \ge 0$, and we must have necessarily $g(y,\zeta) \ge 0$. Since $y \le 0$, this implies $0 \le \zeta \le -y$. We have then showed that $\mathcal{C}(z) \subset [-y,0]$. Consider now the case where $z \in \partial_\ell^+\mathcal{S}$ and take some arbitrary $\zeta \in \mathcal{C}(z) = \mathcal{C}_0(z) \cup \mathcal{C}_1(z)$. If $\zeta \in \mathcal{C}_0(z)$, then similarly as above, we must have $pg(y,\zeta) - k \ge 0$. Since $y > 0$, this implies $-y \le \zeta < 0$. If $\zeta \in \mathcal{C}_1(z)$, i.e. $x - \theta(\zeta,p) - k \ge 0$ and $y + \zeta \ge 0$, and recalling that $x < k$, we must have also $-y \le \zeta < 0$. We have then showed that $\mathcal{C}(z) \subset [-y,0)$.

Notice that for $z \in (\partial_\ell^-\mathcal{S} \cup \partial_\ell^+\mathcal{S}) \cap \mathcal{N}_\ell$, we have $L_0(\Gamma(z,\zeta)) \le p\bar{g}(y) - k < 0$ for all $\zeta \in \mathbb{R}$. Hence, $\mathcal{C}_0(z) = \emptyset$. We have already seen that $\mathcal{C}_1(z) = \{-y\}$ when $z \in \partial_\ell^-\mathcal{S} \cup \partial_\ell^{+,\lambda}\mathcal{S}$ and so $\mathcal{C}(z) = \{-y\}$ when $z \in (\partial_\ell^-\mathcal{S} \cup \partial_\ell^{+,\lambda}\mathcal{S}) \cap \mathcal{N}_\ell$.

Chapter 2

A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact: Numerical Aspect

Joint work with Mohamed MNIF.

Abstract : We investigate numerical aspects of a portfolio selection problem studied in the first chapter, in which we suggest a model of liquidity risk and price impact and formulate the problem as an impulse control problem under state constraint. We show that our impulse control problem could be reduced to an iterative sequence of optimal stopping problems. Given the dimension of our problem and the complexity of its solvency region, we use Monte Carlo methods instead of finite difference methods to calculate the value function, the transaction and no-transaction regions. We provide a numerical approximation algorithm as well as numerical results for the optimal transaction strategy.

Keywords: impulse control problem, Optimal transaction strategy, Monte Carlo method, Malliavin calculus.

2.1 Introduction

In this chapter, we investigate numerical aspects of a portfolio selection problem studied in the first chapter, in which we suggest a model of liquidity risk and price impact. Transactions are allowed only in discrete times and incur some fixed costs. Under the impact of liquidity risk, prices are pushed up when buying stock shares and moved down when selling shares. The investor maximizes his expected utility of terminal liquidation wealth, under a solvency constraint. This problem is formulated as an impulse control problem under state constraint. In the first chapter, we characterize the value function as the unique constrained viscosity solution to the associated Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJBQVI) (1.3.1). We recall that our associated HJBQVI has, in addition to time variable, three variables : x, y, and p, respectively the cash holding, the stock holding, and the stock share price.

Hamilton-Jacobi-Bellman equations are usually solved by using numerical methods based on finite difference methods. The Howard algorithm, which consists in computing two sequences: the optimal strategy and the value function, is known to be efficient for the resolution of these types of equation. From Barles and Souganidis [6], we know that a monotone, stable and consistant scheme insures the convergence of the algorithm to the unique viscosity solution of the HJBQVI. This algorithm has a complexity in $O(N^n)$ where N is the number of points of the grid in one axis and n the dimension of the equation. Chancelier, Øksendal, and Sulem [17] used the Howard algorithm to solve numerically a bi-dimensional HJBQVI related to a problem of optimal consumption and portfolio with both fixed and proportional transaction costs. They solved the problem in a bounded domain and they assumed zero Neumann boundary conditions on the localized boundary. The disadvantage of the finite difference method is its suitability to only solve HJB equations when the solvency region has a simple shape such $\mathbb{R}_+^n$ or when its boundaries are straight. In [17], the solvency region presents some corners. However the authors simplify the problem by omitting the points of the domain where either the number of shares or the amount of money in the portfolio is non-positive.

Korn [47] studied the problem of portfolio optimization with strictly positive transaction costs and impulse control. He presented a sequence of optimal stopping problems where the reward function is expressed in terms of the impulse operator. He proved the convergence of the sequence of optimal stopping problems towards the value function of the initial problem. Chancelier, Øksendal and Sulem [17] suggested an iterative method to solve the impulse control problem. They considered an auxiliary value function where the transactions number is bounded by a positive number.

In this chapter, we prove that both iterative methods coincide. We study numerically our problem by reducing the impulse control problem to an iterative sequence of optimal stopping problems. Then, we introduce a numerical approximation algorithm for every optimal stopping problem based on ideas of Monte Carlo numerical procedure which requires the computation of many conditional expectations. Several methods can be used for the valuation of these regression functions. We choose the Malliavin Calculus based Method suggested by Fournié, Lasry, Lebuchoux, Lions, and Touzi [29] and then developed by Bouchard, Ekeland, and Touzi [10]. Our numerical approach named value-iteration algorithm could be adapted to any shape of the solvency region and we do not need to assume any artificial boundary condition.

The paper is organized as follows. We first show that the value function could be obtained as the limit of an iterative procedure when each step is an optimal stopping problem and the reward function is related to the impulse operator. We then provide a numerical method based on Malliavin calculus and give numerical results for the optimal transaction strategy.

2.2 Convergence of the iterative scheme

We first introduce the following subsets of $\mathcal{A}(t,z)$, the set of the admissible impulse control strategies :

$$\mathcal{A}_n(t,z) \;:=\; \{\alpha \;=\; (\tau_k,\xi_k)_{k=0,\dots,n} \in \mathcal{A}(t,z)\}$$

and the corresponding value function v_n, which describes the value function when the investor is allowed to trade at most n times:

$$v_n(t,z) \;:=\; \sup_{\alpha\in\mathcal{A}_n(t,z)} \mathbb{E}[U_L(Z_T)] \quad (t,z)\in[0,T]\times\overline{\mathcal{S}}. \tag{2.2.1}$$

For $t\in[0,T]$ and $z=(x,y,p)\in\overline{\mathcal{S}}$, if $x,\ y$ are both nonnegative, we clearly have $L(Z_s^{0,t,z})\geq 0$, and so $\mathcal{A}_0(t,z)$ is nonempty. Otherwise, if $x<0,\ y\geq 0$ or $x\geq 0,\ y<0$, due to the diffusion term $P^{0,t,z}$, it is clear that the probability for $L(Z_s^{0,t,z})$ to be negative before time T, is strictly positive, so that $\mathcal{A}_0(t,z)$ is empty. Hence, the value function for $n=0$ is initialized to:

$$v_0(t,z)=\begin{cases}\mathbb{E}\left[U_L(Z_T^{0,t,z})\right] & \text{if } x\geq 0,\ y\geq 0\\ -\infty & \text{otherwise}\end{cases}$$

We now show the convergence of the sequence of the value functions v_n towards our initial value function v.

Lemma 2.2.1 *For all $(t,z)\in\mathcal{S}$*

$$\lim_{n\to\infty} v_n(t,z) \;=\; v(t,z).$$

Proof. From the definition of $\mathcal{A}_n(t,z)$, we have:

$$\mathcal{A}_n(t,z) \;\subset\; \mathcal{A}_{n+1}(t,z) \;\subset\; \mathcal{A}(t,z).$$

As such,

$$v_n(t,z) \;\leq\; v_{n+1}(t,z) \;\leq\; v(t,z),$$

which gives the existence of the limit and the first inequality:

$$\lim_{n\to\infty} v_n(t,z) \;\leq\; v(t,z). \tag{2.2.2}$$

Given $\varepsilon > 0$, from the definition of v, there exists an impulse control $\alpha \;=\; (\tau_1, \tau_2, ...; \xi_1, \xi_2, ...) \in \mathcal{A}(t,z)$ such that

$$\mathbb{E}[U_L(Z_T^\alpha)] \;\geq\; v(t,z) - \varepsilon, \tag{2.2.3}$$

with Z^α diffusing under the impulse control α.
We now set the control

$$\alpha_n \;:=\; (\tau_1, \tau_2, ..., \tau_{n-1}, \underline{\tau}; \xi_1, \xi_2, ..., \xi_{n-1}, y_{\tau_{n-1}}),$$

where $\tau_{n-1} \;<\; \underline{\tau} \;<\; \min\{\tau_n, T\}$. We see that $\alpha_n \in \mathcal{A}_n(t,z)$ and consider the corresponding process $Z^{(\alpha_n)}$. Using Fatou lemma, we obtain:

$$\liminf_{n\to\infty} \mathbb{E}[U_L(Z_T^{(\alpha_n)})] \;\geq\; \mathbb{E}[\liminf_{n\to\infty} U_L(Z_T^{(\alpha_n)})] \;=\; \mathbb{E}[U_L(Z_T^\alpha)] \tag{2.2.4}$$

Using (2.2.3) and (2.2.4), we obtain

$$\liminf_{n\to\infty} v_n(t,z) \;\geq\; \liminf_{n\to\infty} \mathbb{E}[U_L(Z_T^{(\alpha_n)})] \;\geq\; v(t,z) - \varepsilon.$$

As we obtain the latter inequality with an arbitrary $\varepsilon > 0$, and combining with the relation (2.2.2), we obtain the desired result:

$$\lim_{n\to\infty} v_n(t,z) \;=\; v(t,z).$$

Theorem 2.2.1 *We define $\varphi_n(t,z)$ iteratively as a sequence of optimal stopping problems:*

$$\begin{aligned} \varphi_{n+1}(t,z) &= \sup_{\tau\in\mathcal{S}_{t,T}} \mathbb{E}\left[\mathcal{H}\varphi_n(\tau, Z_\tau^{0,t,z})\right], \\ \varphi_0(t,z) &= v_0(t,z), \end{aligned}$$

where $\mathcal{S}_{t,T}$ is the set of stopping times in $[t,T]$. Then

$$\varphi_n(t,z) \;=\; v_n(t,z).$$

Remark 2.2.1 Theorem 2.2.1 together with Lemma 2.2.1 show that

$$\lim_{n\to\infty} \varphi_n(t,z) \;=\; v(t,z), \quad (t,z) \in [0,T] \times \mathcal{S}.$$

so that the iteration scheme for φ_n provides an approximation for v.

Remark 2.2.2 The value function φ_n satisfies the system of variational inequalities, which can be solved by induction starting from φ_0:

$$\min\left[-\frac{\partial \varphi_{n+1}}{\partial t} - \mathcal{L}\varphi_{n+1}\ ,\ \varphi_{n+1} - \mathcal{H}\varphi_n\right] \ = \ 0, \quad (t,z) \in [0,T) \times \mathcal{S},$$

together with the terminal condition:

$$\varphi_{n+1}(T,z) \ = \ \mathcal{H}\varphi_n(T,z).$$

Proof of Theorem 2.2.1. We show by induction that $v_n(t,z) = \varphi_n(t,z)$, for all n. First, we have $v_0 = \varphi_0$. Considering an impulse control strategy $\alpha_1 \ = \ (\tau,\xi) \ \in \ \mathcal{A}_1(t,z)$, we clearly have

$$\begin{aligned} \varphi_1(t,z) &\geq \mathbb{E}[\mathcal{H}\varphi_0(\tau, Z_\tau^{0,t,z})], \\ &\geq \mathbb{E}[\mathcal{H}v_0(\tau, Z_\tau^{0,t,z})]. \end{aligned}$$

From the definition of the operator $\mathcal{H}$, we obtain

$$\varphi_1(t,z) \ \geq \ \mathbb{E}[v_0(\tau, \Gamma(Z_\tau^{0,t,z},\xi))], \quad \forall \alpha_1 = \ (\tau,\xi) \in \mathcal{A}_1(t,z). \tag{2.2.5}$$

Let $Z^{(\alpha_1)}$ be the diffusion of Z, starting at time t, with $Z_t^{(\alpha_1)} = z$, and evolving under the impulse control α_1. Relation (2.2.5) becomes:

$$\varphi_1(t,z) \ \geq \ E[v_0(\tau, Z_\tau^{(\alpha_1)})], \quad \forall\ \alpha_1 = (\tau,\xi) \ \in \ \mathcal{A}_1(t,z). \tag{2.2.6}$$

Given the arbitrariness of α_1 and by using the dynamic programming principle applied to $v_1(t,z)$, we obtain

$$\varphi_1(t,z) \ \geq \ v_1(t,z).$$

From the definition of φ_1, for a given $\varepsilon \ > \ 0$, there exists τ^* such that

$$\varphi_1(t,z) - \varepsilon \ \leq \ \mathbb{E}[\mathcal{H}\varphi_0(\tau^*, Z_{\tau^*}^{0,t,z})]. \tag{2.2.7}$$

From the compactness of the set of admissible transactions, there exists ξ^* such that

$$\begin{aligned} \varphi_1(t,z) - \varepsilon &\leq \mathbb{E}[v_0(\tau^*, \Gamma(Z_{\tau^*}^{0,t,z},\xi^*))], \\ &\leq E[v_0(\tau^*, Z_{\tau^*}^{(*)})], \end{aligned}$$

where $Z^{(*)}$ is the processus starting at time t, with $Z_t^{(*)} = z$, and evolving under the impulse control $\alpha^* := (\tau^*,\xi^*)$.
Using the dynamic programming principle applied on $v_1(t,z)$, we obtain

$$\varphi_1(t,z) - \varepsilon \quad \leq \quad v_1(t,z).$$

The latter inequality is satisfied for any value of $\varepsilon > 0$, as such, we have

$$\varphi_1(t,z) \;\leq\; v_1(t,z),$$

which leads to $\varphi_1(t,z) = v_1(t,z)$, for all $(t,z) \in [0,T) \times \mathcal{S}$.
By induction, assuming that for a given n, we have $\varphi_n(t,z) = v_n(t,z)$, we will prove that $\varphi_{n+1}(t,z) = v_{n+1}(t,z)$. By definition, we have for any $\alpha_{n+1} = (\tau_1,...,\tau_{n+1},\xi_1,...,\xi_{n+1}) \in \mathcal{A}_{n+1}(t,z)$,

$$\begin{aligned}
\varphi_{n+1}(t,z) &\geq \mathbb{E}[\mathcal{H}\varphi_n(\tau_1, Z^{0,t,z}_{\tau_1})], \\
&\geq \mathbb{E}[v_n(\tau_1, \Gamma(Z^{0,t,z}_{\tau_1}, \xi_1))], \\
&\geq \mathbb{E}[v_n(\tau_1, Z^{(n+1)}_{\tau_1})],
\end{aligned} \tag{2.2.8}$$

where $Z^{(n+1)}$ is the diffusion starting at time t, with $Z^{(n+1)}_t = z$ and evolves under the control α_{n+1}. Given the arbitrariness of the control α_{n+1} and by using the dynamic programming principle applied to v_{n+1}, relation (2.2.8) becomes:

$$\varphi_{n+1}(t,z) \;\geq\; v_{n+1}(t,z).$$

To prove the opposite inequality, we use the definition of φ_{n+1}. For any $\varepsilon > 0$, there exists τ^* such that

$$\varphi_{n+1}(t,z) - \varepsilon \;\leq\; \mathbb{E}[\mathcal{H}\varphi_n(\tau^*, Z^{0,t,z}_{\tau^*})], \tag{2.2.9}$$

$$\leq\; \mathbb{E}[\mathcal{H}v_n(\tau^*, Z^{0,t,z}_{\tau^*})]. \tag{2.2.10}$$

From the compactness of the set of admissible transactions, there also exists ξ^* such that

$$\mathcal{H}v_n(\tau^*, Z^{0,t,z}_{\tau^*}) \;=\; v_n(\tau^*, Z^{(\alpha^*)}_{\tau^*}),$$

where $Z^{(\alpha^*)}$, the processus starting at time t, with $Z_t = z$, evolves under the impulse control $\alpha^* := (\tau^*, \xi^*)$. Using the dynamic programming principle applied on v_{n+1}, the relation (2.2.10) becomes

$$\begin{aligned}
\varphi_{n+1}(t,z) - \varepsilon &\leq \mathbb{E}[v_n(\tau^*, Z^{(\alpha^*)}_{\tau^*})], \\
&\leq v_{n+1}(t,z).
\end{aligned}$$

The inequality is obtained for any given ε, this leads to the required inequality

$$\varphi_{n+1}(t,z) \;=\; v_{n+1}(t,z).$$

□

2.3 Numerical study

The objective of this section is the computation of a sequence of optimal stopping problem:

$$v_{n+1}(t,z) \;=\; \sup_{\tau \in \mathcal{S}_{t,T}} E\left[e^{-r(\tau-t)}\mathcal{H}v_n(\tau, X^{0,t,x}_\tau, y, P^{0,t,p}_\tau)\right],\; z \in \bar{\mathcal{S}}$$

and the associated trade region and the no-trade region. We choose the Monte Carlo numerical procedure for the implementation.

2.3.1 The Monte Carlo method

Let $\mathbf{T}_m = \{t_l = lT/m\}_{0\le l\le m}$ be the partition of the time interval $\mathbf{T} = [0,T]$. We denote by h_t the time step $\frac{T}{m}$, and by $\mathcal{S}_{m,t,T}$ the subset of $\mathcal{S}_{t,T}$ defined by

$$\mathcal{S}_{m,t,T} \quad = \quad \{\tau \in \mathcal{S}_{t,T}\ ;\ \tau \in \mathbf{T}_m\}.$$

Let $h_z := (h_x, h_y, h_p) = (1/M_1, 1/M_2, 1/M_3)$, where $(M_1, M_2, M_3) \in \mathbb{N}^{*3}$ denotes the finite difference step in the state coordinate $z = (x, y, p)$. Since the liquidation solvency region is unbounded, we localize $\bar{\mathcal{S}}$ to $D = \{z \in \bar{\mathcal{S}}$ s.t. $-L_1 \le x \le L_1,\ -L_2 \le y \le L_2,\ 0 \le z \le L_3\}$, where L_1, L_2 and L_3 are positive constants. We define the grid:

$$\Omega_{h_z} \quad = \quad \{z = (ih_x, jh_y, kh_p) \in D, -M_1L_1 \le i \le M_1L_1, -M_2L_2 \le j \le M_2L_2, 0 \le k \le M_3L_3\}.$$

For the implementation, we simulate N independent Brownian motions as follows :

$$W_{t_{l+1}} - W_{t_l} \sim N(0, h_t).$$

Then, the price path is given by

$$P^0_{t_{l+1}} \quad = \quad P^0_{t_l} e^{(b-\frac{\sigma^2}{2})h_t + \sigma(W_{t_{l+1}} - W_{t_l})}$$

For the approximation of the value function v_n at the point (t, Z^0_t) where Z^0_t is the random vector (X^0_t, y, Z^0_t) (the randomness is only in the third component of this vector), $t \in \mathbf{T}_m$, two cases are possible :

Case 1: If $Z_t \in [-M_1, M_1] \times [-M_2, M_2] \times [-M_3, M_3]$, then

$$\hat{Z}^0_t = \sum_{i=1}^{N(\Omega_{h_z})} z_i \mathbf{1}_{A_i}(Z^0_t),$$

where $N(\Omega_{h_z}) := Card\{z$ s.t. $z \in \Omega_{h_z}\}$ and $(A_i)_{1\le i\le N(\Omega_{h_z})}$ is a Borel partition of $\bar{\mathcal{S}}$ defined by

$$A_i = \left\{ z \in \bar{\mathcal{S}} \text{ s.t. } |z_i - z| = \min_{1\le j\le N(\Omega_{h_z})} |z_j - z| \right\}.$$

$|.|$ denotes the canonical Euclidean norm, and we take $v_n(t, Z^0_t) \approx v_n(t, \hat{Z}^0_t)$.

Case 2: If $Z_t \notin [-M_1, M_1] \times [-M_2, M_2] \times [-M_3, M_3]$, $|Z^0_t - \hat{Z}^0_t|$ could be large. To approximate $v_n(t, Z^0_t)$, we use the growth condition of the value function

$$v(t,z) \le \frac{e^{\rho(T-t)}}{\gamma}(x + \frac{p}{\lambda}(1 - e^{-\lambda y}))^\gamma \tag{2.3.11}$$

where ρ is a positive constant s.t. $\rho > \frac{\gamma}{1-\gamma}\frac{b^2+r^2+\sigma^2 r(1-\gamma)}{\sigma^2}$ (See Proposition 1.4.1 in Chapter 1).

The approximation of $v_n(t, Z_t^0)$ is given by

$$v_n(t, Z_t^0) \approx v_n(t, \hat{Z}_t^0) \frac{\left(X_t^0 + \frac{P_t^0(1-e^{-\lambda y})}{\lambda}\right)^\gamma}{\left(\hat{X}_t^0 + \frac{\hat{P}_t^0(1-e^{-\lambda y})}{\lambda}\right)^\gamma} \tag{2.3.12}$$

The discrete time approximation for the value function v_n is given by :

$$v_{n+1}(t, z) = \sup_{\tau \in \mathcal{S}_{m,t,T}} E\left[e^{-r(\tau - t)} \mathcal{H} v_n(\tau, \hat{X}_\tau^{0,t,x}, y, \hat{P}_\tau^{0,t,p})\right], \quad (t, z) \in \mathbf{T}_m \times \Omega_{h_z}.$$

The Snell envelop is computed by backward induction :

$$v_{n+1}(t_m, z) = \mathcal{H} v_n(t_m, z)$$

and

$$v_{n+1}(t_{l-1}, z) = \max\left\{\mathcal{H} v_n(t_{l-1}, z); e^{-rh_t} E[v_{n+1}(t_l, Z_{t_l}^0) | \mathcal{F}_{t_{l-1}}]\right\}, \quad 1 \le l \le m,$$

where $\mathcal{F}_{t_{l-1}} = \sigma(P_{t_j}, j \le l-1)$ is the discrete-time filtration. Hence :

$$E[v_{n+1}(t_l, Z_{t_l}^0) | \mathcal{F}_{t_{l-1}}] = E[v_{n+1}(t_l, Z_{t_l}^0) | P_{t_{l-1}}] =: \rho(t_{l-1}, P_{t_{l-1}}^0), \quad 1 \le l \le m.$$

2.3.2 Estimation of the conditional expectation using Malliavin Calculus

Here, we are interested in computing the conditional expectation $E[v_n(t+h, Z_{t+h}^0) | P_t]$. From the definition of v_n, we have $v_n \le v$. The main idea of the Malliavin method consists in using the Malliavin integration by part formula in order to get rid of the Dirac point masses in the following expression :

$$E[v_n(t+h, Z_{t+h}^0) | P_t = p] = \frac{E[v_n(t+h, Z_{t+h}^0) \delta_p(P_t^0)]}{E[\delta_p(P_t^0)]}. \tag{2.3.13}$$

We focus on the calculation of $E[v_n(t+h, Z_{t+h}^0) \delta_p(P_t^0)]$. We recall that $P_t^0 = p_0 e^{(b - \frac{\sigma^2}{2})t + \sigma W_t}$. We now define

$$\hat{v}_{n,t+h,x,y}(Br) := v_n(t+h, x, y, e^{(b-\frac{\sigma^2}{2})(t+h) + \sigma Br}),$$

and

$$\hat{p}_t := \frac{1}{\sigma}(ln \frac{p}{p_0} - (b - \frac{\sigma^2}{2})t).$$

We obtain :

$$E[v_n(t+h, Z_{t+h}^0) \delta_p(P_t^0)] = E[\hat{v}_{n,t+h,X_{t+h}^0,y}(W_{t+h}) \delta_{\hat{p}_t}(W_t)].$$

By the independence of Brownian motion's increments, we have :

$$E[\hat{v}_{n,t+h,X_{t+h},y}(W_{t+h}) \delta_{\hat{p}_t}(W_t)] = \int\int \hat{v}_{n,t+h,X_{t+h},y}(w_1 + w_2) \delta_{\hat{p}_t}(w_1) \varphi(\frac{w_1}{\sqrt{t}}) \varphi(\frac{w_2}{\sqrt{h}}) \frac{dw_1}{\sqrt{t}} \frac{dw_2}{\sqrt{h}},$$

where φ is the density of standard one dimensional normal distribution. Using the growth condition of the value function v (2.3.11) we obtain

$$E[|v_n(t+h, Z^0_{t+h})|^2] < \infty. \tag{2.3.14}$$

Recalling that $\delta_x(w_1)$ is a derivative of $\mathbf{1}_{w_1 \geq x}$, using (2.3.14) and by integration by parts formula with respect to w_1 variable and then with respect to variable w_2, we get :

$$\begin{aligned} & E[\hat{v}_{n,t+h,X_{t+h},y}(W_{t+h})\delta_{\hat{p}_t}(W_t)] \\ = \;& E\left[\hat{v}_{n,t+h,X_{t+h},y}(W_{t+h})\mathbf{1}_{[\hat{p}_t,\infty)}(W_t)\left(-\frac{W_t}{t} + \frac{W_{t+h}-W_t}{h}\right)\right]. \end{aligned} \tag{2.3.15}$$

By denoting $A_h := \frac{W_t}{t} - \frac{W_{t+h}-W_t}{h}$, it follows that :

$$E[v_n(t+h, Z_{t+h})\delta_p(P_t)] \;=\; E\left[v_n(t+h, Z_{t+h})\mathbf{1}_{[p,\infty)}(P_t)A_h\right].$$

2.3.3 Variance reduction by localization

By using Monte Carlo Method, we recover a convergence rate of the order $\sqrt{N}$ for the conditional expectation estimator where N is the simulation number. However, the variance of the estimator explodes as h tends to zero since $\limsup_{h\longrightarrow 0} A_h = \infty$ and $\liminf_{h\longrightarrow 0} A_h = -\infty$. To find a remedy to this problem, we introduce localizing functions. Such functions catch the idea that the relevant information for the computation of $E[g(S_{t+h})|S_t = x]$ is located in the neighborhood of x. Let φ be an arbitrary localizing function. By definition, φ is smooth, bounded and it satisfies $\varphi(0) = 1$. Recalling the same arguments as in (2.3.15) and using (2.3.14), we obtain a family of alternative representations of the conditional expectation given by (2.3.13) :

$$\begin{aligned} E[v_n(t+h, Z^0_{t+h})\delta_p(P^0_t)] &= E\left[v_n(t+h, Z_{t+h})\delta_{\hat{p}_t}(W_t)\varphi(W_t - \hat{p}_t)\right] \\ &= E\left[\mathbf{1}_{W_t > \hat{p}_t}\; v_n(t+h, Z^0_{t+h})(\varphi(W_t - \hat{p}_t)A_h - \varphi'(W_t - \hat{p}_t))\right]. \end{aligned}$$

Moreover, it is possible to reduce the Monte Carlo estimator variance by a convenient choice of the localizing function. We consider the integrated mean square error :

$$J(\varphi) \;:=\; \int_{\mathbb{R}} E\left[\mathbf{1}_{W_t > \hat{p}_t}\; v_n^2(t+h, Z^0_{t+h})A^2_{h,\varphi}\right] dx, \tag{2.3.16}$$

where we adopted the following notation : $A_{h,\varphi} := \varphi(W_t - \hat{p}_t)A_h - \varphi'(W_t - \hat{p}_t)$ and we are interested in minimizing J respect to the subset $\{\ \varphi$ smooth, bounded and $\varphi(0) = 1\}$. Following [10], we prove that the optimal localizing function is given by :

$$\varphi(x) = e^{\nu_h x} \quad where \quad \nu_h := \left(\frac{E[v_n^2(t+h, Z^0_{t+h})A_h^2]}{E[v_n^2(t+h, Z^0_{t+h})]}\right)^{\frac{1}{2}}.$$

In conclusion, we obtain

$$E[v_n(t+h, Z_{t+h})\delta_p(P_t)] \;=\; E\left[v_n(t+h, Z_{t+h})\mathbf{1}_{[p,\infty)}e^{\nu_h(W_t - \hat{p}_t)}(A_h - \nu_h)\right].$$

2.3.4 Algorithm and discrete value function formula

The algorithm computes two sequences $\{v_n, \zeta_n\}_{n\geq 1}$ by performing the following steps.
Parameters: ε, λ, k, L_1, L_2, L_3, N the number simulation, T, M_1, M_2, M_3 and m.
Initialisation: $v_0 = (v_0(t,z))_{(t,z)\in \mathbf{T}_m\times\Omega_{h_z}}$, $n = 0$.
Step 1: Compute $\mathcal{H}v_n$ and ζ_n on $\mathbf{T}_m \times \Omega_{h_z}$ defined by

$$\mathcal{H}v_n(t,z) \quad = \quad \sup_{\zeta\in\hat{C}(z)} v_n(t,\hat{\Gamma}(z,\zeta)),\ (t,z)\in \mathbf{T}_m\times\Omega_{h_z},$$

where $\hat{\Gamma}(z,\zeta) = (\hat{x},\hat{y},\hat{p}) = \sum_{i=1}^{N(\Omega_{h_z})} z_i \mathbf{1}_{A_i}(x-\zeta p e^{\lambda y} - k, y+\zeta, p e^{\lambda\zeta})$, $z_i \in \Omega_{h_z}$ and

$$\hat{C}(z) = \{\zeta\in\mathbb{R} \text{ s.t. } \hat{L}(\hat{\Gamma}(z,\zeta)) := \max\left[\hat{L}_0(z), L_1(z)\right]\mathbf{1}_{y\geq 0} + \hat{L}_0(z)1_{y<0} \geq 0\},$$

$\hat{L}_0(z)$ is the closest point of the grid $(ih_x)_{-M_1L_1\leq i\leq M_1L_1}$ to the point $x - l(y,p) - k$.
Step 2: According to the previous section, we are able to calculate the value function :

$$v_{n+1}(t_l,z) \quad = \quad \max\left\{\mathcal{H}v_n(t_l,z); e^{-rh_t}\hat{\rho}_n(t_l,z)\right\},\ 0\leq l\leq m-1,\ z\in\Omega_{h_z},$$

Let us denote $P^{(i)}$ the i-th price simulation such that $1\leq i\leq N$, where N is the simulation number. Then, we define the estimators of ρ_n by :

$$\tilde{\rho}_n(t_l,z) \quad = \quad \frac{\frac{1}{N}\sum_{i=1}^{N} v_n(t_{l+1}, Z^{0(i)}_{t_{l+1}})\mathbf{1}_{[p,\infty)} e^{\nu_h(W^{(i)}_{t_{l+1}} - \hat{p}^{(i)}_{t_{l+1}})}(A^{(i)}_h - \nu_h)}{\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{[p,\infty)} e^{\nu_h(W^{(i)}_{t_{l+1}} - \hat{p}^{(i)}_{t_{l+1}})}(A^{(i)}_h - \nu_h)},$$

where $z = (x,y,p)$, $A^{(i)}_h := \frac{W^{(i)}_{t_l}}{t_l} - \frac{W^{(i)}_{t_{l+1}} - W^{(i)}_{t_{l+1}}}{h}$, $\hat{p}^i_{t_l} = \frac{1}{\sigma}(ln\frac{p}{p_0} - (b - \frac{\sigma^2}{2})t_l)$ and $W^{(i)}$ i-th simulation of W.
Taking into account the growth condition of the value function, we truncate the estimator $\tilde{\rho}_n$:

$$\hat{\rho}_n(t_l,z) := \tilde{\rho}_n(t_l,z) \wedge \frac{1}{\gamma}e^{\rho(T-t_l)}(x+\frac{p}{\lambda})^\gamma,$$

which improves the algorithm.
Step 3: Stopping test: If $||v_{n+1} - v_n||_\infty \leq \varepsilon$, stop, otherwise go to step 1.

2.3.5 Numerical results

The computation is achieved with a cluster of 13 Intel Xeon Processors running at 2.8 Ghz with 2 Giga Bytes of RAM. Numerical tests are performed with the following numerical constants

$$\gamma = 0.5,\ r = 0.1,\ \alpha = 0.12,\ \sigma = 0.3.$$

We take

$$L_1 = L_2 = L_3 = 10,\ T = 1,\ h_x = 1,\ h_y = 0.5,\ h_z = 1,\ h_t = 0.1.$$

According to Bouchard and Touzi [11], in order to achieve an error estimate of the order of $n^{-1/2}$, we have to choose a number N of simulated trajectories such that $N = O(n^{7/2})$. Contrary to the policy iteration algorithm (named Howard algorithm), the value-iteration algorithm needs more iterations to converge. This explains that several days are necessary to achieve the whole computation. We equally mention that by using the probabilistic approach, we do not need to assume any boundary condition as in [17]. However when the trajectories are outside our bounded domain, we approximate the value function by taking into account the growth condition (see (2.3.12)).

A partition of the solvency region $\mathcal{S}$ is displayed in figures (2.1)-(2.2)-(2.3) for different values of P and λ. It consists of three regions: Buy (B), Sell (S), and No-Trade (NT) regions. The domain between $R1$ and $R2$ corresponds to the region reached by the state variable after a purchase or a sale of risky asset, dictated by the optimal strategy. Due to the presence of fixed costs, the lines $R1$ and $R2$ do not coincide with $D1$ and $D2$ boundaries of the no-transaction region.

We equally try to see the sensitivity of different parameters and variables.

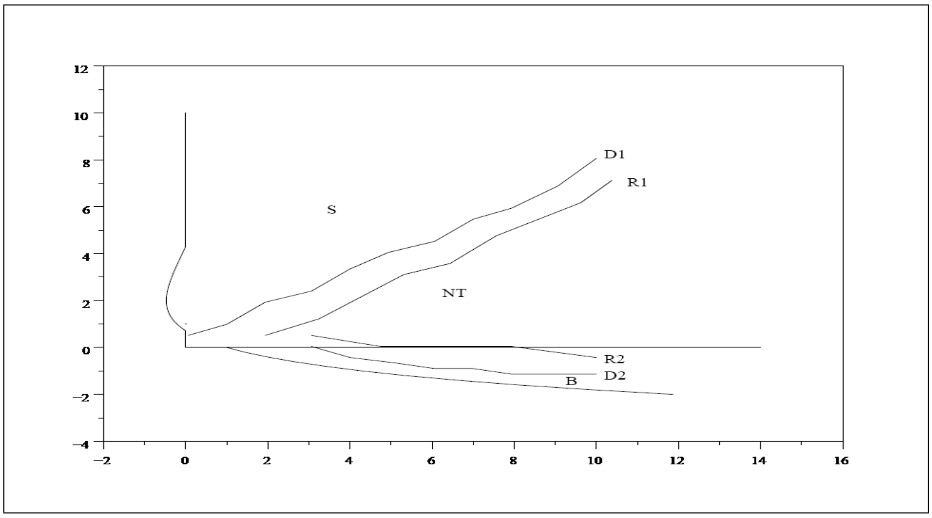


Figure 2.1: The optimal transaction policy for p=2, λ=0.5 and k=1

$\star$ First, there is a reduction in the No-Trade region when the price of the risky asset P increases, i.e. the line $D1$ moves downwards while the line $D2$ marginally moves upwards (see Figures (2.1)-(2.2)). The interpretation of this observation is the following :

- in the case where the investor has a significant long position in the risky asset, he is required to reduce his risky asset position when the share price goes up. This phenomenon has also been observed in the Merton model [53].

- in the case where the investor has a significant short position in the risky asset, he is required to buy back shares in order to reduce the risk when the share price goes up.

⋆ Second, we look at the impact of the coefficient of the impact price λ. We notice that when λ increases, the NT region widens (see Figures (2.2)-(2.3)). In particular, the line $D1$ significantly moves upwards. Economically, it means that when the liquidity impact increases, the investor should trade less frequently.

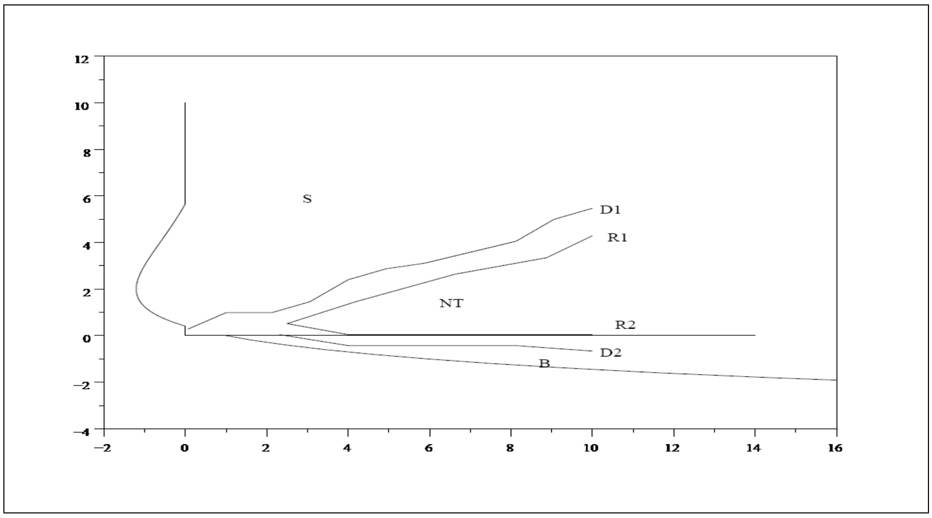


Figure 2.2: The optimal transaction policy for p=3, λ=0.5 and k=1

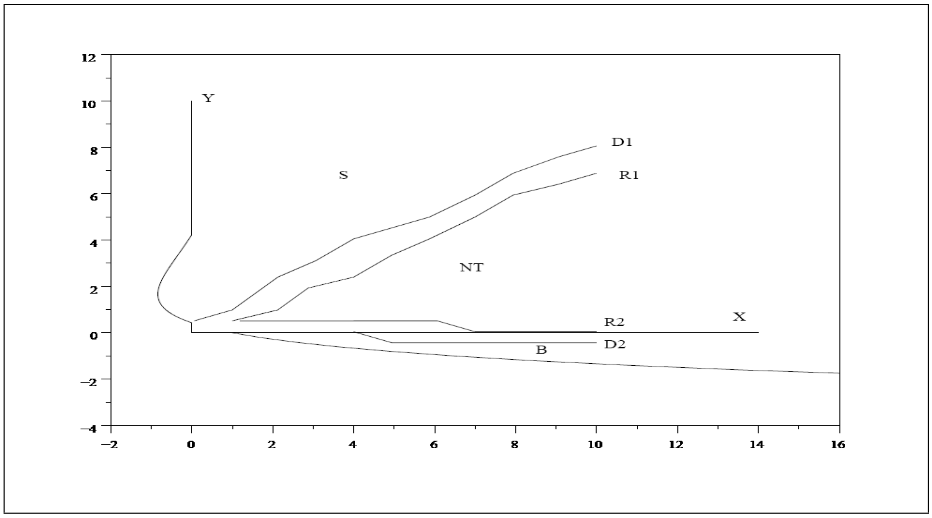


Figure 2.3: The optimal transaction policy for p=3, λ=0.6 and k=1

Part II

STOCHASTIC CONTROL: REAL OPTIONS

Chapter 3

Explicit solution to an optimal switching problem in the two-regime case

Joint paper with Huyên PHAM, to appear in *SIAM Journal on Control and Optimization.*

Abstract : This paper considers the problem of determining the optimal sequence of stopping times for a diffusion process subject to regime switching decisions. This is motivated in the economics literature, by the investment problem under uncertainty for a multi-activity firm involving opening and closing decisions. We use a viscosity solutions approach combined with the smooth-fit property, and explicitly solve the problem in the two regime case when the state process is of geometric Brownian nature. The results of our analysis take several qualitatively different forms, depending on model parameter values.

Keywords: Optimal switching, system of variational inequalities, viscosity solutions, smooth-fit principle.

3.1 Introduction

The theory of optimal stopping and its generalization, thoroughly studied in the seventies, have received a renewed interest with a variety of applications in economics and finance. These applications range from asset pricing (American options, swing options) to firm investment and real options. We refer to [26] for a classical and well documented reference on the subject.

In this paper, we consider the optimal switching problem for a one dimensional stochastic process X. The diffusion process X may take a finite number of regimes that are switched at stopping time decisions. For example in the firm's investment problem under uncertainty, a company (oil tanker, electricity station ....) manages several production activities operating in different modes or regimes representing a number of different economic outlooks (e.g. state of economic growth, open or closed production activity, ...). The process X is the price of input or output goods of the firm and its dynamics may differ according to the regimes. The firm's project yields a running payoff that depends on the commodity price X and on the regime choice. The transition from one regime to another one is realized sequentially at time decisions and incurs certain fixed costs. The problem is to find the switching strategy that maximizes the expected value of profits resulting from the project.

Optimal switching problems were studied by several authors, see [7] or [64]. These control problems lead via the dynamic programming principle to a system of variational inequalities. Applications to option pricing, real options and investment under uncertainty were considered by [12], [27], [37], and [35]. In this last paper, the drift and volatility of the state process depend on an uncontrolled finite-state Markov chain, and the author provides an explicit solution to the optimal stopping problem with applications to Russian options. In [37], the authors solve a two-regime (operating and closed) switching problem. Their approach consists in using the notions of Backward SDE and Snell envelope to prove the existence of an optimal strategy as well as providing its expression. In [12], an explicit solution is found for a resource extraction problem with two regimes (open or closed field), a linear profit function and a price process following a geometric Brownian motion. In [27], a similar model is solved with a general profit function in one regime and equal to zero in the other regime. In both models [12], [27], there is no switching in the diffusion process : changes of regimes only affect the payoff functions. Their method of resolution is to construct a solution to the dynamic programming system by guessing a priori the form of the strategy, and then validate a posteriori the optimality of their candidate by a verification argument.

Our model combines regime switchings both on the diffusion process and on the general profit functions. We use a viscosity solutions approach for determining the solution to the system of variational inequalities. In particular, we derive directly the smooth-fit property of the value functions and the structure of the switching regions. Explicit solutions are provided in the following cases :

⋆ the drift and volatility terms of the diffusion take two different regime values, and the profit functions are identical of power type,

⋆ there is no switching on the diffusion process, and the two different profit functions satisfy a general condition, including typically power functions.

We also consider the cases for which both switching costs are positive, and for which one of the two is negative. This last case is interesting in applications where a firm chooses between an open or closed activity, and may regain a fraction of its opening costs when it decides to close. The results of our analysis take several qualitatively different forms, depending on model parameter values, essentially the payoff functions and the switching costs.

The paper is organized as follows. We formulate in Section 3.2 the optimal switching problem. In Section 3.3, we state the system of variational inequalities satisfied by the value functions in the viscosity sense. The smooth-fit property for this problem, proved in [57], plays an important role in our subsequent analysis. We also state some useful properties on the switching regions. In Section 3.4, we explicitly solve the problem in the two-regime case when the state process is of geometric Brownian nature.

3.2 Formulation of the optimal switching problem

We consider a stochastic system that can operate in d modes or regimes. The regimes can be switched at a sequence of stopping times decided by the operator (individual, firm, ...). The indicator of the regimes is modelled by a cadlag process I_t valued in $\mathbb{I}_d = \{1, \ldots, d\}$. The stochastic system X (commodity price, salary, ...) is valued in $\mathbb{R}_+^* = (0, \infty)$ and satisfies the s.d.e.

$$dX_t = b_{I_t} X_t dt + \sigma_{I_t} X_t dW_t, \tag{3.2.1}$$

where W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions. $b_i \in \mathbb{R}$, and $\sigma_i > 0$ are the drift and volatility of the system X once in regime $I_t = i$ at time t.

A strategy decision for the operator is an impulse control α consisting of a double sequence $\tau_1, \ldots, \tau_n, \ldots, \kappa_1, \ldots, \kappa_n, \ldots$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where τ_n are stopping times, $\tau_n < \tau_{n+1}$ and $\tau_n \to \infty$ a.s., representing the switching regimes time decisions, and κ_n are $\mathcal{F}_{\tau_n}$-measurable valued in $\mathbb{I}_d$, and representing the new value of the regime at time $t = \tau_n$. We denote by $\mathcal{A}$ the set of all such impulse controls. Now, for any initial condition $(x, i) \in (0, \infty) \times \mathbb{I}_d$, and any control $\alpha = (\tau_n, \kappa_n)_{n \geq 1} \in \mathcal{A}$, there exists a unique strong solution valued in $(0, \infty) \times \mathbb{I}_d$ to the controlled stochastic system :

$$X_0 = x, \quad I_{0^-} = i, \tag{3.2.2}$$

$$dX_t = b_{\kappa_n} X_t dt + \sigma_{\kappa_n} X_t dW_t, \quad I_t = \kappa_n, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0. \tag{3.2.3}$$

Here, we set $\tau_0 = 0$ and $\kappa_0 = i$. We denote by $(X^{x,i}, I^i)$ this solution (as usual, we omit the dependence in α for notational simplicity). We notice that $X^{x,i}$ is a continuous process and I^i is a cadlag process, possibly with a jump at time 0 if $\tau_1 = 0$ and so $I_0 = \kappa_1$.

We are given a running profit function $f : \mathbb{R}_+ \times \mathbb{I}_d \to \mathbb{R}$ and we set $f_i(.) = f(., i)$ for $i \in \mathbb{I}_d$. We assume that for each $i \in \mathbb{I}_d$, the function f_i is nonnegative and is Hölder continuous on $\mathbb{R}_+$: there exists $\gamma_i \in (0, 1]$ s.t.

$$|f_i(x) - f_i(\hat{x})| \quad \leq \quad C|x - \hat{x}|^{\gamma_i}, \quad \forall x, \hat{x} \in \mathbb{R}_+, \tag{3.2.4}$$

for some positive constant C. Without loss of generality (see Remark 3.2.1), we may assume that $f_i(0) = 0$. We also assume that for all $i \in \mathbb{I}_d$, the conjugate of f_i is finite on $(0, \infty)$:

$$\tilde{f}_i(y) \quad := \quad \sup_{x \geq 0} [f_i(x) - xy] \; < \; \infty, \quad \forall y > 0. \tag{3.2.5}$$

The cost for switching from regime i to j is a constant equal to g_{ij}, with the convention $g_{ii} = 0$, and we assume the triangular condition :

$$g_{ik} \quad < \quad g_{ij} + g_{jk}, \quad j \neq i, k. \tag{3.2.6}$$

This last condition means that it is less expensive to switch directly in one step from regime i to k than in two steps via an intermediate regime j. Notice that a switching cost g_{ij} may be negative, and condition (3.2.6) for $i = k$ prevents arbitrage by switching back and forth, i.e.

$$g_{ij} + g_{ji} \quad > \quad 0, \quad i \neq j \in \mathbb{I}_d. \tag{3.2.7}$$

The expected total profit of running the system when initial state is (x, i) and using the impulse control $\alpha = (\tau_n, \kappa_n)_{n \geq 1} \in \mathcal{A}$ is

$$J_i(x, \alpha) \quad = \quad E\left[\int_0^\infty e^{-rt} f(X_t^{x,i}, I_t^i)dt - \sum_{n=1}^\infty e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}\right].$$

Here $r > 0$ is a positive discount factor, and we use the convention that $e^{-r\tau_n(\omega)} = 0$ when $\tau_n(\omega) = \infty$. We also make the standing assumption :

$$r \quad > \quad b := \max_{i \in \mathbb{I}_d} b_i. \tag{3.2.8}$$

The objective is to maximize this expected total profit over all strategies α. Accordingly, we define the value functions

$$v_i(x) \quad = \quad \sup_{\alpha \in \mathcal{A}} J_i(x, \alpha), \quad x \in \mathbb{R}_+^*, \; i \in \mathbb{I}_d. \tag{3.2.9}$$

We shall see in the next section that under (3.2.5) and (3.2.8), the expectation defining $J_i(x)$ is well-defined and the value function v_i is finite.

Remark 3.2.1 The initial values $f_i(0)$ of the running profit functions received by the firm manager (the controller) before any decision are considered as included into the switching costs when changing of regime. This means that w.l.o.g. we may assume that $f_i(0) = 0$. Indeed, for any profit function f_i, and by setting $\bar{f}_i = f_i - f_i(0)$, we have for all $x > 0$, $\alpha \in \mathcal{A}$,

$$
\begin{aligned}
J_i(x,\alpha) &= E\left[\sum_{n=1}^{\infty}\int_{\tau_{n-1}}^{\tau_n} e^{-rt} f(X_t^{x,i},\kappa_{n-1})dt - \sum_{n=1}^{\infty} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}\right] \\
&= E\left[\sum_{n=1}^{\infty}\int_{\tau_{n-1}}^{\tau_n} e^{-rt}\left(\bar{f}(X_t^{x,i},\kappa_{n-1}) + f_{\kappa_{n-1}}(0)\right)dt - \sum_{n=1}^{\infty} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}\right] \\
&= E\left[\sum_{n=1}^{\infty}\int_{\tau_{n-1}}^{\tau_n} e^{-rt}\bar{f}(X_t^{x,i},\kappa_{n-1})dt + \frac{f_{\kappa_0}(0)}{r}\right. \\
&\qquad \left. - \sum_{n=1}^{\infty} e^{-r\tau_n}\left(g_{\kappa_{n-1},\kappa_n} + \frac{f_{\kappa_n}(0) - f_{\kappa_{n-1}}(0)}{r}\right)\right] \\
&= \frac{f_i(0)}{r} + E\left[\int_0^{\infty} e^{-rt}\bar{f}(X_t^{x,i}, I_t^i)dt - \sum_{n=1}^{\infty} e^{-r\tau_n}\tilde{g}_{\kappa_{n-1},\kappa_n}\right],
\end{aligned}
$$

with modified switching costs that take into account the possibly different initial values of the profit functions :

$$
\tilde{g}_{ij} = g_{ij} + \frac{f_j(0) - f_i(0)}{r}.
$$

3.3 System of variational inequalities, switching regions and viscosity solutions

We first state the linear growth property and the boundary condition on the value functions.

Lemma 3.3.1 *We have for all $i \in \mathbb{I}_d$:*

$$
\max_{j\in\mathbb{I}_d}[-g_{ij}] \le v_i(x) \le \frac{xy}{r-b} + \max_{j\in\mathbb{I}_d}\frac{\tilde{f}_j(y)}{r} + \max_{j\in\mathbb{I}_d}[-g_{ij}], \ \forall x > 0, \ \forall y > 0. \quad (3.3.1)
$$

In particular, we have $v_i(0^+) = \max_{j\in\mathbb{I}_d}[-g_{ij}]$.

Proof. By considering the particular strategy $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\kappa}_n)$ of immediate switching from the initial state (x,i) to state (x,j), $j \in \mathbb{I}_d$ (eventually equal to i), at cost g_{ij} and then doing nothing, i.e. $\tilde{\tau}_1 = 0$, $\tilde{\kappa}_1 = j$, $\tilde{\tau}_n = \infty$, $\tilde{\kappa}_n = j$ for all $n \ge 2$, we have

$$
J_i(x,\tilde{\alpha}) = E\Big[\int_0^{\infty} e^{-rt} f_j(\tilde{X}_t^{x,j})dt - g_{ij}\Big],
$$

where $\tilde{X}^{x,j}$ denotes the geometric brownian in regime j starting from x at time 0. Since f_j is nonnegative, and by the arbitrariness of j, we get the lower bound in (3.3.1).

Given an initial state $(X_0, I_{0^-}) = (x,i)$ and an arbitrary impulse control $\alpha = (\tau_n, \kappa_n)$, we get from the dynamics (3.2.2)-(3.2.3), the following explicit expression of $X^{x,i}$:

$$\begin{aligned} X_t^{x,i} &= xY_t(i) \\ &:= x\left(\prod_{l=0}^{n-1} e^{b_{\kappa_l}(\tau_{l+1}-\tau_l)} Z^{\kappa_l}_{\tau_l,\tau_{l+1}}\right) e^{b_{\kappa_n}(t-\tau_n)} Z^{\kappa_n}_{\tau_n,t}, \quad \tau_n \le t < \tau_{n+1}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.3.2}$$

where

$$Z^j_{s,t} = \exp\left(\sigma_j(W_t - W_s) - \frac{\sigma_j^2}{2}(t-s)\right), \quad 0 \le s \le t, \quad j \in \mathbb{I}_d. \tag{3.3.3}$$

Here, we used the convention that $\tau_0 = 0$, $\kappa_0 = i$, and the product term from l to $n-1$ in (3.3.2) is equal to 1 when $n = 1$. We then deduce the inequality $X_t^{x,i} \le xe^{bt}M_t$, for all t, where

$$M_t = \left(\prod_{l=0}^{n-1} Z^{\kappa_l}_{\tau_l,\tau_{l+1}}\right) Z^{\kappa_n}_{\tau_n,t}, \quad \tau_n \le t < \tau_{n+1}, \quad n \in \mathbb{N}. \tag{3.3.4}$$

Now, we notice that (M_t) is a martingale obtained by continuously patching the martingales $(Z^{\kappa_{n-1}}_{\tau_{n-1},t})$ and $(Z^{\kappa_n}_{\tau_n,t})$ at the stopping times τ_n, $n \ge 1$. In particular, we have $E[M_t] = M_0$ $= 1$ for all t.

We set $\tilde f(y) = \max_{j\in\mathbb{I}_d} \tilde f_i(y)$, $y > 0$, and we notice by definition of $\tilde f_i$ in (3.2.5) that $f(X_t^{x,i}, I_t^i) \le yX_t^{x,i} + \tilde f(y)$ for all t, y. Moreover, we show by induction on N that for all N ≥ 1, $\tau_1 \le \ldots \le \tau_N$, $\kappa_0 = i$, $\kappa_n \in \mathbb{I}_d$, $n = 1,\ldots,N$:

$$-\sum_{n=1}^{N} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n} \le \max_{j\in\mathbb{I}_d}[-g_{ij}], \quad a.s.$$

Indeed, the above assertion is obviously true for $N = 1$. Suppose now it holds true at step N. Then, at step $N+1$, we distinguish two cases : If $g_{\kappa_N,\kappa_{N+1}} \ge 0$, then we have $-\sum_{n=1}^{N+1} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n} \le -\sum_{n=1}^{N} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}$ and we conclude by the induction hypothesis at step N. If $g_{\kappa_N,\kappa_{N+1}} < 0$, then by (3.2.6), and since $\tau_N \le \tau_{N+1}$, we have $-e^{-r\tau_N} g_{\kappa_{N-1},\kappa_N} - e^{-r\tau_{N+1}} g_{\kappa_N,\kappa_{N+1}} \le e^{-r\tau_N} g_{\kappa_{N-1},\kappa_{N+1}}$, and so $-\sum_{n=1}^{N+1} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n} \le -\sum_{n=1}^{N} e^{-r\tau_n} g_{\tilde\kappa_{n-1},\tilde\kappa_n}$, with $\tilde\kappa_n = \kappa_n$ for $n = 1,\ldots,N-1$, $\tilde\kappa_N = \kappa_{N+1}$. We then conclude by the induction hypothesis at step N.

It follows that

$$\begin{aligned} J_i(x,\alpha) &\le E\left[\int_0^\infty e^{-rt}\left(yxe^{bt}M_t + \tilde f(y)\right)dt + \max_{j\in\mathbb{I}_d}[-g_{ij}]\right] \\ &= \int_0^\infty e^{-(r-b)t} yxE[M_t]dt + \int_0^\infty e^{-rt}\tilde f(y)dt + \max_{j\in\mathbb{I}_d}[-g_{ij}] \\ &= \frac{xy}{r-b} + \frac{\tilde f(y)}{r} + \max_{j\in\mathbb{I}_d}[-g_{ij}]. \end{aligned}$$

From the arbitrariness of α, this shows the upper bound for v_i.

By sending x to zero and then y to infinity into the r.h.s. of (3.3.1), and recalling that $\tilde{f}_i(\infty) = f_i(0) = 0$ for $i \in \mathbb{I}_d$, we conclude that v_i goes to $\max_{j\in\mathbb{I}_d}[-g_{ij}]$ when x tends to zero. □

We next show the Hölder continuity of the value functions.

Lemma 3.3.2 *For all $i \in \mathbb{I}_d$, v_i is Hölder continuous on $(0,\infty)$:*

$$|v_i(x) - v_i(\hat{x})| \;\; \leq \;\; C|x-\hat{x}|^\gamma, \quad \forall x, \hat{x} \in (0,\infty), \quad \textit{with} \;\; |x-\hat{x}| \leq 1,$$

for some positive constant C, and where $\gamma = \min_{i\in\mathbb{I}_d}\gamma_i$ of condition (3.2.4).

Proof. By definition (3.2.9) of v_i and under condition (3.2.4), we have for all $x, \hat{x} \in (0,\infty)$, with $|x-\hat{x}| \leq 1$:

$$\begin{aligned} |v_i(x) - v_i(\hat{x})| \;\; &\leq \;\; \sup_{\alpha\in\mathcal{A}} |J_i(x,\alpha) - J_i(\hat{x},\alpha)| \\ &\leq \;\; \sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-rt}\left|f(X_t^{x,i}, I_t^i) - f(X_t^{\hat{x},i}, I_t^i)\right| dt\right] \\ &\leq \;\; C\sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-rt}\left|X_t^{x,i} - X_t^{\hat{x},i}\right|^{\gamma_{I_t^i}} dt\right] \\ &= \;\; C\sup_{\alpha\in\mathcal{A}}\int_0^\infty E\left[e^{-rt}|x-\hat{x}|^{\gamma_{I_t^i}}|Y_t(i)|^{\gamma_{I_t^i}}dt\right] \\ &\leq \;\; C|x-\hat{x}|^\gamma \sup_{\alpha\in\mathcal{A}}\int_0^\infty e^{-(r-b)t}E|M_t|^{\gamma_{I_t^i}}dt \end{aligned} \tag{3.3.5}$$

by (3.3.2) and (3.3.4). For any $\alpha = (\tau_n, \kappa_n)_n \in \mathcal{A}$, by the independence of $(Z^{\kappa_n}_{\tau_n,\tau_{n+1}})_n$ in (3.3.3), and since

$$E\left[\left|Z^{\kappa_n}_{\tau_n,\tau_{n+1}}\right|^{\gamma_{\kappa_n}}\Big|\mathcal{F}_{\tau_n}\right] \;\; = \;\; E\left[\exp\left(\gamma_{\kappa_n}(\gamma_{\kappa_n}-1)\frac{\sigma^2_{\kappa_n}}{2}(\tau_{n+1}-\tau_n)\right)\Big|\mathcal{F}_{\tau_n}\right] \;\leq\; 1, \quad a.s.,$$

we clearly see that $E|M_t|^{\gamma_{I_t^i}} \leq 1$ for all $t \geq 0$. We thus conclude with (3.3.5). □

The dynamic programming principle combined with the notion of viscosity solutions are known to be a general and powerful tool for characterizing the value function of a stochastic control problem via a PDE representation, see [28]. We recall the definition of viscosity solutions for a P.D.E in the form

$$H(x, v, D_x v, D^2_{xx} v) \;\; = \;\; 0, \quad x \in \mathcal{O}, \tag{3.3.6}$$

where $\mathcal{O}$ is an open subset in $\mathbb{R}^n$ and H is a continuous function and non-increasing in its last argument (with respect to the order of symmetric matrices).

Definition 3.3.1 *Let v be a continuous function on $\mathcal{O}$. We say that v is a viscosity solution to* (3.3.6) *on $\mathcal{O}$ if it is*

(i) a viscosity supersolution to (3.3.6) *on $\mathcal{O}$: for any $\bar{x} \in \mathcal{O}$ and any C^2 function φ in a neighborhood of $\bar{x}$ s.t. $\bar{x}$ is a local minimum of $v - \varphi$, we have :*

$$H(\bar{x}, v(\bar{x}), D_x\varphi(\bar{x}), D^2_{xx}\varphi(\bar{x})) \quad \geq \quad 0.$$

and

(ii) a viscosity subsolution to (3.3.6) *on $\mathcal{O}$: for any $\bar{x} \in \mathcal{O}$ and any C^2 function φ in a neighborhood of $\bar{x}$ s.t. $\bar{x}$ is a local maximum of $v - \varphi$, we have :*

$$H(\bar{x}, v(\bar{x}), D_x\varphi(\bar{x}), D^2_{xx}\varphi(\bar{x})) \quad \leq \quad 0.$$

Remark 3.3.1 **1.** By misuse of notation, we shall say that v is viscosity supersolution (resp. subsolution) to (3.3.6) by writing :

$$H(x, v, D_x v, D^2_{xx} v) \quad \geq \text{ (resp. } \leq) \quad 0, \quad x \in \mathcal{O}, \tag{3.3.7}$$

2. We recall that if v is a smooth C^2 function on $\mathcal{O}$, supersolution (resp. subsolution) in the classical sense to (3.3.7), then v is a viscosity supersolution (resp. subsolution) to (3.3.7).
3. There is an equivalent formulation of viscosity solutions, which is useful for proving uniqueness results, see [19] :
(i) A continuous function v on $\mathcal{O}$ is a viscosity supersolution to (3.3.6) if

$$H(x, v(x), p, M) \quad \geq \quad 0, \quad \forall x \in \mathcal{O}, \ \forall (p, M) \in J^{2,-}v(x).$$

(ii) A continuous function v on $\mathcal{O}$ is a viscosity subsolution to (3.3.6) if

$$H(x, v(x), p, M) \quad \leq \quad 0, \quad \forall x \in \mathcal{O}, \ \forall (p, M) \in J^{2,+}v(x).$$

Here $J^{2,+}v(x)$ is the second order superjet defined by :

$$J^{2,+}v(x) \quad = \quad \left\{ (p, M) \in \mathbb{R}^n \times S^n \ : \ \limsup_{\substack{x' \to x \\ x \in \mathcal{O}}} \frac{v(x') - v(x) - p.(x' - x) - \frac{1}{2}(x' - x).M(x' - x)}{|x' - x|^2} \ \leq \ 0 \right\},$$

S^n is the set of symmetric $n \times n$ matrices, and $J^{2,-}v(x) = -J^{2,+}(-v)(x)$.

In the sequel, we shall denote by $\mathcal{L}_i$ the second order operator associated to the diffusion X when we are in regime i : for any C^2 function φ on $(0, \infty)$,

$$\mathcal{L}_i\varphi \quad = \quad \frac{1}{2}\sigma_i^2 x^2 \varphi'' + b_i x \varphi'.$$

We then have the following PDE characterization of the value functions v_i by means of viscosity solutions.

Theorem 3.3.1 *The value functions v_i, $i \in \mathbb{I}_d$, are the unique viscosity solutions with linear growth condition on $(0,\infty)$ and boundary condition $v_i(0^+) = \max_{j\in\mathbb{I}_d}[-g_{ij}]$ to the system of variational inequalities :*

$$\min\left\{rv_i - \mathcal{L}_i v_i - f_i\ ,\ v_i - \max_{j\neq i}(v_j - g_{ij})\right\} \;=\; 0, \quad x \in (0,\infty), \quad i \in \mathbb{I}_d. \qquad (3.3.8)$$

This means
(1) Viscosity property : for each $i \in \mathbb{I}_d$, v_i is a viscosity solution to

$$\min\left\{rv_i - \mathcal{L}_i v_i - f_i\ ,\ v_i - \max_{j\neq i}(v_j - g_{ij})\right\} \;=\; 0, \quad x \in (0,\infty). \qquad (3.3.9)$$

(2) Uniqueness property : if w_i, $i \in \mathbb{I}_d$, are viscosity solutions with linear growth conditions on $(0,\infty)$ and boundary conditions $w_i(0^+) = \max_{j\in\mathbb{I}_d}[-g_{ij}]$ to the system of variational inequalities (3.3.8) *, then $v_i = w_i$ on $(0,\infty)$.*

Proof. (1) The viscosity property follows from the dynamic programming principle and is proved in [57].
(2) Uniqueness results for switching problems has been proved in [64] in the finite horizon case under different conditions. For sake of completeness, we provide, in Appendix, a proof of comparison principle in our infinite horizon context, which implies the uniqueness result. □

Remark 3.3.2 For fixed $i \in \mathbb{I}_d$, we also have uniqueness of viscosity solution to equation (3.3.9) in the class of continuous functions with linear growth condition on $(0,\infty)$ and given boundary condition on 0. In the next section, we shall use either uniqueness of viscosity solutions to the system (3.3.8) or for fixed i to equation (3.3.9), for the identification of an explicit solution in the two-regime case $d = 2$.

We shall also combine the uniqueness result for the viscosity solutions with the smooth-fit property on the value functions that we state below.

For any regime $i \in \mathbb{I}_d$, we introduce the switching region :

$$\mathcal{S}_i \;=\; \left\{x \in (0,\infty)\ : v_i(x) = \max_{j\neq i}(v_j - g_{ij})(x)\right\}.$$

$\mathcal{S}_i$ is a closed subset of $(0,\infty)$ and corresponds to the region where it is optimal for the operator to change of regime. The complement set $\mathcal{C}_i$ of $\mathcal{S}_i$ in $(0,\infty)$ is the so-called continuation region :

$$\mathcal{C}_i \;=\; \left\{x \in (0,\infty)\ : v_i(x) > \max_{j\neq i}(v_j - g_{ij})(x)\right\},$$

where the operator remains in regime i. In this open domain, the value function v_i is smooth C^2 on $\mathcal{C}_i$ and satisfies in a classical sense :

$$rv_i(x) - \mathcal{L}_i v_i(x) - f_i(x) \;=\; 0, \quad x \in \mathcal{C}_i.$$

As a consequence of the condition (3.2.6), we have the following elementary partition property of the switching regions, see Lemma 4.2 in [57] :

$$\mathcal{S}_i = \cup_{j \neq i} \mathcal{S}_{ij}, \quad i \in \mathbb{I}_d,$$

where

$$\mathcal{S}_{ij} = \{x \in \mathcal{C}_j \ : v_i(x) = (v_j - g_{ij})(x)\}.$$

$\mathcal{S}_{ij}$ represents the region where it is optimal to switch from regime i to regime j and stay here for a moment, i.e. without changing instantaneously from regime j to another regime. The following Lemma gives some partial information about the structure of the switching regions.

Lemma 3.3.3 *For all $i \neq j$ in $\mathbb{I}_d$, we have*

$$\mathcal{S}_{ij} \subset Q_{ij} := \{x \in \mathcal{C}_j \ : (\mathcal{L}_j - \mathcal{L}_i) v_j(x) + (f_j - f_i)(x) - r g_{ij} \geq 0\}.$$

Proof. Let $x \in \mathcal{S}_{ij}$. By setting $\varphi_j = v_j - g_{ij}$, this means that x is a minimum of $v_i - \varphi_j$ with $v_i(x) = \varphi_j(x)$. Moreover, since x lies in the open set $\mathcal{C}_j$ where v_j is smooth, we have that φ_j is C^2 in a neighborhood of x. By the supersolution viscosity property of v_i to the PDE (3.3.8), this yields :

$$r\varphi_j(x) - \mathcal{L}_i \varphi_j(x) - f_i(x) \geq 0. \tag{3.3.10}$$

Now recall that for $x \in \mathcal{C}_j$, we have

$$r v_j(x) - \mathcal{L}_j v_j(x) - f_j(x) = 0,$$

so that by substituting into (3.3.10), we obtain :

$$(\mathcal{L}_j - \mathcal{L}_i) v_j(x) + (f_j - f_i)(x) - r g_{ij} \geq 0,$$

which is the required result. □

We quote the smooth fit property on the value functions, proved in [57].

Theorem 3.3.2 *For all $i \in \mathbb{I}_d$, the value function v_i is continuously differentiable on $(0, \infty)$.*

Remark 3.3.3 In a given regime i, the variational inequality satisfied by the value function v_i is a free-boundary problem as in optimal stopping problem, which divides the state space into the switching region (stopping region in pure optimal stopping problem) and the continuation region. The main difficulty with regard to optimal stopping problems for proving the smooth-fit property through the boundaries of the switching regions, comes from the fact that the switching region for the value function v_i depends also on the other value functions v_j. The method in [57] use viscosity solutions arguments and the condition of one-dimensional state space is critical for proving the smooth-fit property. The crucial conditions in this paper require that the diffusion coefficient in any regime of the system X is strictly positive on the interior the the state space, which is the case here since $\sigma_i > 0$ for all $i \in \mathbb{I}_d$, and a triangular condition (3.2.6) on the switching costs. Under these conditions, on a point x of the switching region $\mathcal{S}_i$ for regime i, there exists some $j \neq i$ s.t. $x \in \mathcal{S}_{ij}$, i.e. $v_i(x) = v_j(x) - g_{ij}$, and the C^1 property of the value functions is written as : $v_i'(x) = v_j'(x)$ since g_{ij} is constant.

The next result provides suitable conditions for determining a viscosity solution to the variational inequality type arising in our switching problem.

Lemma 3.3.4 *Fix $i \in \mathbb{I}_d$. Let $\mathcal{C}$ be an open set in $(0,\infty)$, $\mathcal{S} = (0,\infty) \setminus \mathcal{C}$ supposed to be an union of a finite number of closed intervals in $(0,\infty)$, and w, h two continuous functions on $(0,\infty)$, with $w = h$ on $\mathcal{S}$ such that*

$$w \quad \text{is } C^1 \quad \text{on } \partial\mathcal{S} \tag{3.3.11}$$

$$w \quad \geq \quad h \quad \text{on } \mathcal{C}, \tag{3.3.12}$$

w is C^2 on $\mathcal{C}$, solution to

$$rw - \mathcal{L}_i w - f_i \quad = \quad 0 \quad \text{on } \mathcal{C}, \tag{3.3.13}$$

and w is a viscosity supersolution to

$$rw - \mathcal{L}_i w - f_i \quad \geq \quad 0 \quad \text{on } \operatorname{int}(\mathcal{S}). \tag{3.3.14}$$

Here $\operatorname{int}(\mathcal{S})$ is the interior of $\mathcal{S}$ and $\partial\mathcal{S} = \mathcal{S} \setminus \operatorname{int}(\mathcal{S})$ its boundary. Then, w is a viscosity solution to

$$\min\{rw - \mathcal{L}_i w - f_i, w - h\} \quad = \quad 0 \quad \text{on } (0,\infty). \tag{3.3.15}$$

Proof. Take some $\bar{x} \in (0,\infty)$ and distinguish the following cases :

$\star$ $\bar{x} \in \mathcal{C}$. Since $w = v$ is C^2 on $\mathcal{C}$ and satisfies $rw(\bar{x}) - \mathcal{L}_i w(\bar{x}) - f_i(\bar{x}) = 0$ by (3.3.13), and recalling $w(\bar{x}) \geq h(\bar{x})$ by (3.3.12), we obtain the classical solution property, and so a fortiori the viscosity solution property (3.3.15) of w at $\bar{x}$.

$\star$ $\bar{x} \in \mathcal{S}$. Then $w(\bar{x}) = h(\bar{x})$ and the viscosity subsolution property is trivial at $\bar{x}$. It remains to show the viscosity supersolution property at $\bar{x}$. If $\bar{x} \in \operatorname{int}(\mathcal{S})$, this follows directly from (3.3.14). Suppose now $\bar{x} \in \partial\mathcal{S}$, and to fix the idea, we consider that $\bar{x}$ is on the left-boundary of $\mathcal{S}$ so that from the assumption on the form of $\mathcal{S}$, there exists $\varepsilon > 0$ s.t. $(\bar{x} - \varepsilon, \bar{x}) \subset \mathcal{C}$ on which w is smooth C^2 (the same argument holds true when $\bar{x}$ is on the right-boundary of $\mathcal{S}$). Take some smooth C^2 function φ s.t. $\bar{x}$ is a local minimum of $w - \varphi$. Since w is C^1 by (3.3.11), we have $\varphi'(\bar{x}) = w'(\bar{x})$. We may also assume w.l.o.g (by taking

ε small enough) that $(w - \varphi)(\bar{x}) \leq (w - \varphi)(x)$ for $x \in (\bar{x} - \varepsilon, \bar{x})$. Moreover, by Taylor's formula, we have :

$$w(\bar{x} - \eta) \;=\; w(\bar{x}) - \eta \int_0^1 w'(\bar{x} - t\eta)dt, \qquad \varphi(\bar{x} - \eta) \;=\; \varphi(\bar{x}) - \eta \int_0^1 \varphi'(\bar{x} - t\eta)dt,$$

so that

$$\int_0^1 \varphi'(\bar{x} - t\eta) - w'(\bar{x} - t\eta)\, dt \;\;\geq\;\; 0, \quad \forall\, 0 < \eta < \varepsilon.$$

Since $\varphi'(\bar{x}) = w'(\bar{x})$, this last inequality is written as

$$\int_0^1 \frac{\varphi'(\bar{x} - t\eta) - \varphi'(\bar{x})}{\eta} - \frac{w'(\bar{x} - t\eta) - w'(\bar{x})}{\eta}\, dt \;\;\geq\;\; 0, \quad \forall\, 0 < \eta < \varepsilon, \tag{3.3.16}$$

Now, from (3.3.13), we have $rw(x) - \mathcal{L}_i w(x) - f_i(x) = 0$ for $x \in (\bar{x} - \varepsilon, \bar{x})$. By sending x towards $\bar{x}$ into this last equality, this shows that $w''(\bar{x}^-) = \lim_{x \nearrow \bar{x}} w''(x)$ exists, and

$$rw(\bar{x}) - b_i \bar{x} w'(\bar{x}) - \frac{1}{2}\sigma_i^2 \bar{x}^2 w''(\bar{x}^-) - f_i(\bar{x}) \;\;=\;\; 0. \tag{3.3.17}$$

Moreover, by sending η to zero into (3.3.16), we obtain :

$$\int_0^1 t[-\varphi''(\bar{x}) + w''(\bar{x}^-)]dt \;\;\geq\;\; 0,$$

and so $\varphi''(\bar{x}) \leq w''(\bar{x}^-)$. By substituting into (3.3.17), and recalling that $w'(\bar{x}) = \varphi'(\bar{x})$, we then obtain :

$$rw(\bar{x}) - \mathcal{L}_i \varphi(\bar{x}) - f_i(\bar{x}) \;\;\geq\;\; 0,$$

which is the required supersolution inequality, and ends the proof. □

Remark 3.3.4 Since $w = h$ on $\mathcal{S}$, relation (3.3.14) means equivalently that h is a viscosity supersolution to

$$rh - \mathcal{L}_i h - f_i \;\;\geq\;\; 0 \quad \text{on } \operatorname{int}(\mathcal{S}). \tag{3.3.18}$$

Practically, Lemma 3.3.4 shall be used as follows in the next section : we consider two C^1 functions v and h on $(0, \infty)$ s.t.

$$\begin{aligned} v(x) \;=\; h(x),\; v'(x) \;&=\; h'(x), \quad x \in \partial\mathcal{S} \\ v \;&\geq\; h \quad \text{on } \mathcal{C}, \end{aligned}$$

v is C^2 on $\mathcal{C}$, solution to

$$rv - \mathcal{L}_i v - f_i \;\;=\;\; 0 \quad \text{on } \mathcal{C},$$

and h is a viscosity supersolution to (3.3.18). Then, the function w defined on $(0,\infty)$ by :

$$w(x) \quad = \quad \begin{cases} v(x), & x \in \mathcal{C} \\ h(x), & x \in \mathcal{S} \end{cases}$$

satisfies the conditions of Lemma 3.3.4 and is so a viscosity solution to (3.3.15). This Lemma combined with uniqueness viscosity solution result may be viewed as an alternative to the classical verification approach in the identification of the value function. Moreover, with our viscosity solutions approach, we shall see in subsection 3.4.2 that Lemma 3.3.3 and smooth-fit property of the value functions in Theorem 3.3.2 provide a direct derivation for the structure of the switching regions and then of the solution to our problem.

3.4 Explicit solution in the two-regime case

In this section, we consider the case where $d = 2$. In this two-regime case, we know from Theorem 3.3.1 that the value functions v_i, $i = 1,2$, are the unique continuous viscosity solutions with linear growth condition on $(0,\infty)$, and boundary conditions $v_i(0^+) = (-g_{ij})_+ := \max(-g_{ij}, 0)$, $j \neq i$, to the system :

$$\min\{rv_1 - \mathcal{L}_1 v_1 - f_1, v_1 - (v_2 - g_{12})\} \quad = \quad 0 \tag{3.4.1}$$

$$\min\{rv_2 - \mathcal{L}_2 v_2 - f_2, v_2 - (v_1 - g_{21})\} \quad = \quad 0. \tag{3.4.2}$$

Moreover, the switching regions are :

$$\mathcal{S}_i \;=\; \mathcal{S}_{ij} \quad = \quad \{x > 0\ : v_i(x) = v_j(x) - g_{ij}\}, \quad i,j = 1,2,\ i \neq j.$$

We set

$$\underline{x}_i^* \;=\; \inf \mathcal{S}_i \;\in\; [0,\infty] \qquad \bar{x}_i^* \;=\; \sup \mathcal{S}_i \;\in\; [0,\infty],$$

with the usual convention that $\inf \emptyset = \infty$.

Let us also introduce some other notations. We consider the second order o.d.e for $i = 1,2$:

$$rv - \mathcal{L}_i v - f_i \quad = \quad 0, \tag{3.4.3}$$

whose general solution (without second member f_i) is given by :

$$v(x) \quad = \quad Ax^{m_i^+} + Bx^{m_i^-},$$

for some constants A, B, and where

$$m_i^- \quad = \quad -\frac{b_i}{\sigma_i^2} + \frac{1}{2} - \sqrt{\left(-\frac{b_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}} \;<\; 0$$

$$m_i^+ \quad = \quad -\frac{b_i}{\sigma_i^2} + \frac{1}{2} + \sqrt{\left(-\frac{b_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}} \;>\; 1.$$

We also denote

$$\hat{V}_i(x) = E\left[\int_0^\infty e^{-rt} f_i(\hat{X}_t^{x,i})dt\right],$$

with $\hat{X}^{x,i}$ the solution to the s.d.e. $d\hat{X}_t = b_i\hat{X}_t dt + \sigma_i\hat{X}_t dW_t$, $\hat{X}_0 = x$. Actually, $\hat{V}_i$ is a particular solution to ode (3.4.3), with boundary condition $\hat{V}_i(0^+) = f_i(0) = 0$. It corresponds to the reward function associated to the no switching strategy from initial state (x,i), and so $\hat{V}_i \leq v_i$.

Remark 3.4.1 If $g_{ij} > 0$, then from (3.2.7), we have $v_i(0^+) = 0 > (-g_{ji})_+ - g_{ij} = v_j(0^+) - g_{ij}$. Therefore, by continuity of the value functions on $(0,\infty)$, we get $\underline{x}_i^* > 0$.

We now give the explicit solution to our problem in the following two situations :
⋆ the diffusion operators are different and the running profit functions are identical.
⋆ the diffusion operators are identical and the running profit functions are different

We also consider the cases for which both switching costs are positive, and for which one of the two is negative, the other being then positive according to (3.2.7). This last case is interesting in applications where a firm chooses between an open or closed activity, and may regain a fraction of its opening costs when it decides to close.

3.4.1 Identical profit functions with different diffusion operators

In this subsection, we suppose that the running functions are identical in the form :

$$f_1(x) = f_2(x) = x^\gamma, \quad 0 < \gamma < 1, \tag{3.4.4}$$

and the diffusion operators are different. A straightforward calculation shows that under (3.4.4), we have

$$\hat{V}_i(x) = K_i x^\gamma, \quad \text{with } K_i = \frac{1}{r - b_i\gamma + \frac{1}{2}\sigma_i^2\gamma(1-\gamma)} > 0, \quad i = 1,2.$$

We show that the structure of the switching regions depends actually only on the sign of $K_2 - K_1$, and of the sign of the switching costs g_{12} and g_{21}. More precisely, we have the following explicit result.

Theorem 3.4.1 *Let i, $j = 1,2$, $i \neq j$.*
1) If $K_i = K_j$, then

$$v_i(x) = \hat{V}_i(x) + (-g_{ij})_+, \qquad x \in (0,\infty),$$
$$\mathcal{S}_i = \begin{cases} \emptyset & \text{if } g_{ij} > 0 \\ (0,\infty) & \text{if } g_{ij} \leq 0. \end{cases}$$

It is always optimal to switch from regime i to j if the corresponding switching cost is non-positive, and never optimal to switch otherwise.

2) If $K_j > K_i$, then we have the following situations depending on the switching costs :
a) $g_{ij} \leq 0$: we have $\mathcal{S}_i = (0,\infty)$, $\mathcal{S}_j = \emptyset$, and

$$v_i = \hat{V}_j - g_{ij}, \qquad v_j = \hat{V}_j.$$

b) $g_{ij} > 0$:

- *if $g_{ji} \geq 0$, then $\mathcal{S}_i = [\underline{x}_i^*,\infty)$ with $\underline{x}_i^* \in (0,\infty)$, $\mathcal{S}_j = \emptyset$, and*

$$v_i(x) = \begin{cases} Ax^{m_i^+} + \hat{V}_i(x), & x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases} \tag{3.4.5}$$

$$v_j(x) = \hat{V}_j(x), \quad x \in (0,\infty) \tag{3.4.6}$$

where the constants A and $\underline{x}_i^*$ are determined by the continuity and smooth-fit conditions of v_i at $\underline{x}_i^*$, and explicitly given by :

$$\underline{x}_i^* = \left(\frac{m_i^+}{m_i^+ - \gamma}\frac{g_{ij}}{K_j - K_i}\right)^{\frac{1}{\gamma}} \tag{3.4.7}$$

$$A = (K_j - K_i)\frac{\gamma}{m_i^+}(\underline{x}_i^*)^{\gamma - m_i^+}. \tag{3.4.8}$$

When we are in regime i,it is optimal to switch to regime j whenever the state process X exceeds the threshold $\underline{x}_i^*$, while when we are in regime j, it is optimal never to switch.

- *if $g_{ji} < 0$, then $\mathcal{S}_i = [\underline{x}_i^*,\infty)$ with $\underline{x}_i^* \in (0,\infty)$, $\mathcal{S}_j = (0,\bar{x}_j^*]$, and*

$$v_i(x) = \begin{cases} Ax^{m_i^+} + \hat{V}_i(x), & x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases} \tag{3.4.9}$$

$$v_j(x) = \begin{cases} v_i(x) - g_{ji}, & x \leq \bar{x}_j^* \\ Bx^{m_j^-} + \hat{V}_j(x), & x > \bar{x}_i^* \end{cases} \tag{3.4.10}$$

where the constants A, B and $\bar{x}_j^* < \underline{x}_i^*$ are determined by the continuity and smooth-fit conditions of v_i and v_j at $\underline{x}_i^*$ and $\bar{x}_j^*$, and explicitly given by :

$$\overline{x}_j = \left[\frac{-m_j^-(g_{ji} + g_{ij}y^{m_i^+})}{(K_i - K_j)(\gamma - m_j^-)(1 - y^{m_i^+ - \gamma})}\right]^{\frac{1}{\gamma}}$$

$$\underline{x}_i = \frac{\overline{x}_j}{y}$$

$$B = \frac{(K_i - K_j)(m_i^+ - \gamma)\underline{x}_i^{\gamma - m_j^-} + m_i^+ g_{ij}\underline{x}_i^{-m_j}}{m_i^+ - m_j^-}$$

$$A = B\underline{x}_i^{m_j^- - m_i^+} - (K_i - K_j)\underline{x}_i^{\gamma - m_i^+} - g_{ij}\underline{x}_i^{-m_i^+}$$

with y solution in $\left(0, \left(-\frac{g_{ji}}{g_{ij}}\right)^{\frac{1}{m_i^+}}\right)$ to the equation :

$$\begin{aligned} & m_i^+(\gamma - m_j^-)\left(1 - y^{m_i^+ - \gamma}\right)\left(g_{ij} y^{m_j^-} + g_{ji}\right) \\ + & \, m_j^-(m_i^+ - \gamma)\left(1 - y^{m_j^- - \gamma}\right)\left(g_{ij} y^{m_i^+} + g_{ji}\right) \quad = 0 \end{aligned}$$

When we are in regime i, it is optimal to switch to regime j whenever the state process X exceeds the threshold $\underline{x}_i^*$, while when we are in regime j, it is optimal to switch to regime i for values of the state process X under the threshold $\bar{x}_j^*$.

Economic interpretation.
In the particular case where $\sigma_1 = \sigma_2$, then $K_2 - K_1 > 0$ means that regime 2 provides a higher expected return b_2 than the one b_1 of regime 1 for the same volatility coefficient σ_i. Moreover, if the switching cost g_{21} from regime 2 to regime 1 is nonnegative, it is intuitively clear that one has always interest to stay in regime 2, which is formalized by the property that $\mathcal{S}_2 = \emptyset$. However, if one receives some gain compensation to switch from regime 2 to regime 1, i.e. the corresponding cost g_{21} is negative, then one has interest to change of regime for small values of the current state. This is formalized by the property that $\mathcal{S}_2 = (0, \bar{x}_2^*]$. On the other hand, in regime 1, one has interest to switch to regime 2, for all current values of the state if the corresponding switching cost g_{12} is non-positive, or from a certain threshold $\underline{x}_1^*$ if the switching cost g_{12} is positive. A similar interpretation holds when $b_1 = b_2$, and $K_2 - K_1 > 0$, i.e. $\sigma_2 < \sigma_1$. Theorem 3.4.1 extends these results for general coefficients b_i and σ_i, and show that the critical parameter value determining the form of the optimal strategy is given by the sign of $K_2 - K_1$ and the switching costs. The different optimal strategy structures are depicted in Figure I.

Proof of Theorem 3.4.1.
1) If $K_i = K_j$, then $\hat{V}_i = \hat{V}_j$. We consider the smooth functions $w_i = \hat{V}_i + (-g_{ij})_+$ for $i, j = 1, 2$ and $j \neq i$. Since $\hat{V}_i$ are solution to (3.4.3), we see that w_i satisfy :

$$rw_i - \mathcal{L}w_i - f_i \quad = \quad r(-g_{ij})_+ \tag{3.4.11}$$

$$w_i - (w_j - g_{ij}) \quad = \quad g_{ij} + (-g_{ij})_+ - (-g_{ji})_+. \tag{3.4.12}$$

Notice that the l.h.s of (3.4.11) and (3.4.12) are both nonnegative by (3.2.7). Moreover, if $g_{ij} > 0$, then the l.h.s. of (3.4.11) is zero, and if $g_{ij} \leq 0$, then $g_{ji} > 0$ and the l.h.s. of (3.4.12) is zero. Therefore, w_i, $i = 1, 2$ is solution to the system :

$$\min\left\{rw_i - \mathcal{L}_i w_i - f_i, w_i - (w_j - g_{ij})\right\} \quad = \quad 0.$$

Since $\hat{V}_i(0^+) = 0$, we have $w_i(0^+) = (-g_{ij})_+$. Moreover, w_i satisfy like $\hat{V}_i$ a linear growth condition. Therefore, from uniqueness of solution to the PDE system (3.4.1)-(3.4.2), we deduce that $v_i = w_i$. As observed above, if $g_{ij} \leq 0$, then the l.h.s. of (3.4.12) is zero, and so $\mathcal{S}_i = (0, \infty)$. Finally, if $g_{ij} > 0$, then the l.h.s. of (3.4.12) is positive, and so $\mathcal{S}_i = \emptyset$.

Figure I

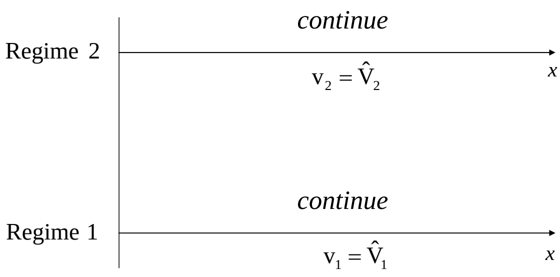


Figure I.1.a: $f_1 = f_2,\ K_1 = K_2,\ g_{12} > 0,\ g_{21} > 0$

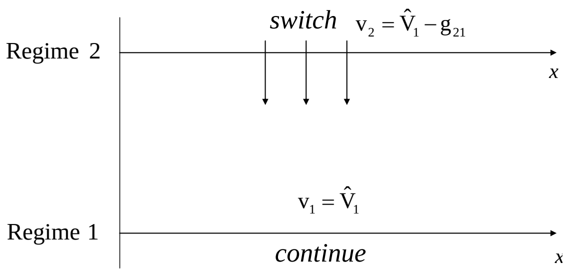


Figure I.1.b: $f_1 = f_2,\ K_1 = K_2,\ g_{12} > 0,\ g_{21} \le 0$

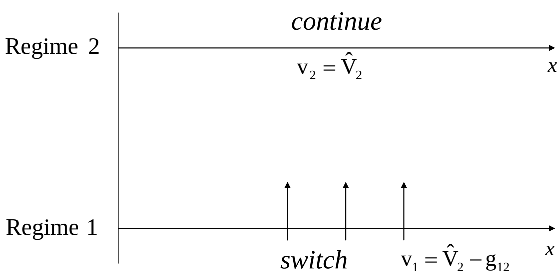


Figure I.2.a: $f_1 = f_2,\ K_2 > K_1,\ g_{12} \le 0$

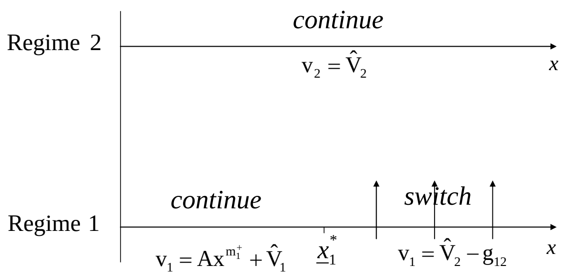


Figure I.2.bi: $f_1 = f_2,\ K_2 > K_1,\ g_{12} > 0,\ g_{21} \ge 0$

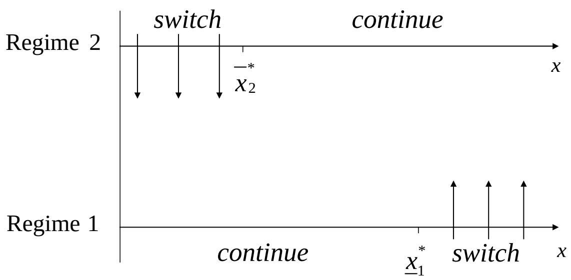


Figure I.2.bii: $f_1 = f_2,\ K_2 > K_1,\ g_{12} > 0,\ g_{21} < 0$

2) We now suppose w.l.o.g. that $K_2 > K_1$.

a) Consider first the case where $g_{12} \leq 0$, and so $g_{21} > 0$. We set $w_1 = \hat{V}_2 - g_{12}$ and $w_2 = \hat{V}_2$. Then, by construction, we have $w_1 = w_2 - g_{12}$ on $(0, \infty)$, and by definition of $\hat{V}_1$ and $\hat{V}_2$:

$$rw_1(x) - \mathcal{L}_1 w_1(x) - f_1(x) = \frac{K_2 - K_1}{K_1} x^\gamma - rg_{12} > 0, \quad \forall\, x > 0.$$

On the other hand, we also have $rw_2 - \mathcal{L}_2 w_2 - f_2 = 0$ on $(0, \infty)$, and $w_2 \geq w_1 - g_{21}$ since $g_{12} + g_{21} \geq 0$. Hence, w_1 and w_2 are smooth (hence viscosity) solutions to the system (3.4.1)-(3.4.2), with linear growth conditions and boundary conditions $w_1(0^+) = V_1(0^+) - g_{12} = (-g_{12})_+$, $w_2(0^+) = \hat{V}_2(0^+) = 0 = (-g_{21})_+$. By uniqueness result of Theorem 3.3.1, we deduce that $v_1 = w_1$, $v_2 = w_2$, and thus $\mathcal{S}_1 = (0, \infty)$, $\mathcal{S}_2 = \emptyset$.

b) Consider now the case where $g_{12} > 0$. We already know from Remark 3.4.1 that $\underline{x}_1^* > 0$, and we claim that $\underline{x}_1^* < \infty$. Otherwise, v_1 should be equal to $\hat{V}_1$. Since $v_1 \geq v_2 - g_{12} \geq \hat{V}_2 - g_{12}$, this would imply $(\hat{V}_2 - \hat{V}_1)(x) = (K_2 - K_1)x^\gamma \leq g_{12}$ for all $x > 0$, an obvious contradiction. By definition of $\underline{x}_1^*$, we have $(0, \underline{x}_1^*) \subset \mathcal{C}_1$. We shall prove actually the equality : $(0, \underline{x}_1^*) = \mathcal{C}_1$, i.e. $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$. On the other hand, the form of $\mathcal{S}_2$ will depend on the sign of g_{21}.

• *Case* : $g_{21} \geq 0$.

We shall prove that $\mathcal{C}_2 = (0, \infty)$, i.e. $\mathcal{S}_2 = \emptyset$. To this end, let us consider the function

$$w_1(x) = \begin{cases} Ax^{m_1^+} + \hat{V}_1(x), & 0 < x < x_1 \\ \hat{V}_2(x) - g_{12}, & x \geq x_1, \end{cases}$$

where the positive constants A and x_1 satisfy

$$Ax_1^{m_1^+} + \hat{V}_1(x_1) = \hat{V}_2(x_1) - g_{12} \tag{3.4.13}$$

$$Am_1^+ x_1^{m_1^+ - 1} + \hat{V}_1'(x_1) = \hat{V}_2'(x_1), \tag{3.4.14}$$

and are explicitly determined by :

$$(K_2 - K_1)x_1^\gamma = \frac{m_1^+}{m_1^+ - \gamma} g_{12} \tag{3.4.15}$$

$$A = (K_2 - K_1)\frac{\gamma}{m_1^+} x_1^{\gamma - m_1^+}. \tag{3.4.16}$$

Notice that by construction, w_1 is C^2 on $(0, x_1) \cup (x_1, \infty)$, and C^1 on x_1.

$\star$ By using Lemma 3.3.4, we now show that w_1 is a viscosity solution to

$$\min\left\{rw_1 - \mathcal{L}_1 w_1 - f_1, w_1 - (\hat{V}_2 - g_{12})\right\} = 0, \quad \text{on } (0, \infty). \tag{3.4.17}$$

We first check that

$$w_1(x) \geq \hat{V}_2(x) - g_{12}, \quad \forall\, 0 < x < x_1, \tag{3.4.18}$$

i.e.

$$G(x) \;:=\; Ax^{m_1^+} + \hat{V}_1(x) - \hat{V}_2(x) + g_{12} \;\geq\; 0, \qquad \forall\, 0 < x < x_1.$$

Since $A > 0$, $0 < \gamma < 1 < m_1^+$, $K_2 - K_1 > 0$, a direct derivation shows that the second derivative of G is positive, i.e. G is strictly convex. By (3.4.14), we have $G'(x_1) = 0$ and so G' is negative, i.e. G is strictly decreasing on $(0, x_1)$. Now, by (3.4.13), we have $G(x_1) = 0$ and thus G is positive on $(0, x_1)$, which proves (3.4.18).

By definition of w_1 on $(0, x_1)$, we have in the classical sense

$$rw_1 - \mathcal{L}_1 w_1 - f_1 \;=\; 0, \quad \text{on } (0, x_1). \tag{3.4.19}$$

We now check that

$$rw_1 - \mathcal{L}_1 w_1 - f_1 \;\geq\; 0, \quad \text{on } (x_1, \infty), \tag{3.4.20}$$

holds true in the classical sense, and so a fortiori in the viscosity sense. By definition of w_1 on (x_1, ∞), and K_1, we have for all $x > x_1$,

$$rw_1(x) - \mathcal{L}_1 w_1(x) - f_1(x) \;=\; \frac{K_2 - K_1}{K_1} x^\gamma - rg_{12}, \quad \forall x > x_1,$$

so that (3.4.20) is satisfied iff $\frac{K_2-K_1}{K_1} x_1^\gamma - rg_{12} \geq 0$ or equivalently by (3.4.15) :

$$\frac{m_1^+}{m_1^+ - \gamma} \;\geq\; rK_1 \;=\; \frac{r}{r - b_1\gamma + \frac{1}{2}\sigma_1^2\gamma(1-\gamma)} \tag{3.4.21}$$

Now, since $\gamma < 1 < m_1^+$, and by definition of m_1^+, we have

$$\frac{1}{2}\sigma_1^2 m_1^+(\gamma - 1) \;<\; \frac{1}{2}\sigma_1^2 m_1^+(m_1^+ - 1) \;=\; r - b_1 m_1^+,$$

which proves (3.4.21) and thus (3.4.20).

Relations (3.4.13)-(3.4.14), (3.4.18)-(3.4.19)-(3.4.20) mean that conditions of Lemma 3.3.4 are satisfied with $\mathcal{C} = (0, x_1)$, $h = \hat{V}_2 - g_{12}$, and we thus get the required assertion (3.4.17).

$\star$ On the other hand, we check that

$$\hat{V}_2(x) \;\geq\; w_1(x) - g_{21}, \qquad \forall x > 0, \tag{3.4.22}$$

which amounts to show

$$H(x) \;:=\; Ax^{m_1^+} + \hat{V}_1(x) - \hat{V}_2(x) - g_{21} \;\leq\; 0, \qquad \forall\, 0 < x < x_1.$$

Since $A > 0$, $0 < \gamma < 1 < m_1^+$, $K_2 - K_1 > 0$, a direct derivation shows that the second derivative of H is positive, i.e. H is strictly convex. By (3.4.14), we have $H'(x_1) = 0$ and so H' is negative, i.e. H is strictly decreasing on $(0, x_1)$. Now, we have $H(0) = -g_{21} \leq 0$

and thus H is negative on $(0,x_1)$, which proves (3.4.22). Recalling that $\hat{V}_2$ is solution to $r\hat{V}_2 - \mathcal{L}_2\hat{V}_2 - f_2 = 0$ on $(0,\infty)$, we deduce obviously from (3.4.22) that $\hat{V}_2$ is a classical, hence a viscosity solution to :

$$\min\Big\{r\hat{V}_2 - \mathcal{L}_2\hat{V}_2 - f_2, \hat{V}_2 - (w_1 - g_{21})\Big\} \;=\; 0, \quad \text{on } (0,\infty). \tag{3.4.23}$$

$\star$ Since $w_1(0^+) = 0 = (-g_{12})_+$, $\hat{V}_2(0^+) = 0 = (-g_{21})_+$, and w_1, $\hat{V}_2$ satisfy a linear growth condition, we deduce from (3.4.17), (3.4.23), and uniqueness to the PDE system (3.4.1)-(3.4.2), that

$$v_1 \;=\; w_1, \qquad v_2 \;=\; \hat{V}_2, \quad \text{on } (0,\infty).$$

This proves $\underline{x}_1^* = x_1$, $\mathcal{S}_1 = [x_1,\infty)$ and $\mathcal{S}_2 = \emptyset$.

• *Case* : $g_{21} < 0$.

We shall prove that $\mathcal{S}_2 = (0,\bar{x}_2^*]$. To this end, let us consider the functions

$$w_1(x) \;=\; \begin{cases} Ax^{m_1^+} + \hat{V}_1(x), & x < \underline{x}_1 \\ w_2(x) - g_{12}, & x \geq \underline{x}_1 \end{cases}$$
$$w_2(x) \;=\; \begin{cases} w_1(x) - g_{21}, & x \leq \bar{x}_2 \\ Bx^{m_2^-} + \hat{V}_2(x), & x > \bar{x}_2, \end{cases}$$

where the positive constants A, B, $\underline{x}_1 > \bar{x}_2$, solution to

$$A\underline{x}_1^{m_1^+} + \hat{V}_1(\underline{x}_1) \;=\; w_2(\underline{x}_1) - g_{12} \;=\; B\underline{x}_1^{m_2^-} + \hat{V}_2(\underline{x}_1) - g_{12} \tag{3.4.24}$$
$$Am_1^+\underline{x}_1^{m_1^+-1} + \hat{V}_1'(\underline{x}_1) \;=\; w_2'(\underline{x}_1) \;=\; Bm_2^-\underline{x}_1^{m_2^--1} + \hat{V}_2'(\underline{x}_1) \tag{3.4.25}$$
$$A\bar{x}_2^{m_1^+} + \hat{V}_1(\bar{x}_2) - g_{21} \;=\; w_1(\bar{x}_2) - g_{21} \;=\; B\bar{x}_2^{m_2^-} + \hat{V}_2(\bar{x}_2) \tag{3.4.26}$$
$$Am_1^+\bar{x}_2^{m_1^+-1} + \hat{V}_1'(\bar{x}_2) \;=\; w_1'(\bar{x}_2) \;=\; Bm_2^-\bar{x}_2^{m_2^--1} + \hat{V}_2'(\bar{x}_2), \tag{3.4.27}$$

exist and are explicitly determined after some calculations by

$$\overline{x}_2 \;=\; \left[\frac{-m_2^-(g_{21} + g_{12}y^{m_1^+})}{(K_1 - K_2)(\gamma - m_2^-)(1 - y^{m_1^+-\gamma})}\right]^{\frac{1}{\gamma}} \tag{3.4.28}$$
$$\underline{x}_1 \;=\; \frac{\overline{x}_2}{y} \tag{3.4.29}$$
$$B \;=\; \frac{(K_1 - K_2)(m_1^+ - \gamma)\underline{x}_1^{\gamma - m_2^-} + m_1^+ g_{12}\underline{x}_1^{-m_2}}{m_1^+ - m_2^-} \tag{3.4.30}$$
$$A \;=\; B\underline{x}_1^{m_2^- - m_1^+} - (K_1 - K_2)\underline{x}_1^{\gamma - m_1^+} - g_{12}\underline{x}_1^{-m_1^+}, \tag{3.4.31}$$

with y solution in $\left(0, \left(-\frac{g_{21}}{g_{12}}\right)^{\frac{1}{m_1^+}}\right)$ to the equation :

$$\begin{aligned} & m_1^+(\gamma - m_2^-)\left(1 - y^{m_1^+-\gamma}\right)\left(g_{12}y^{m_2^-} + g_{21}\right) \\ & + m_2^-(m_1^+ - \gamma)\left(1 - y^{m_2^--\gamma}\right)\left(g_{12}y^{m_1^+} + g_{21}\right) \;=\; 0. \end{aligned} \tag{3.4.32}$$

Using (3.2.7), we have $y < \left(-\frac{g_{21}}{g_{12}}\right)^{\frac{1}{m_1^+}} < 1$. As such, $0 < \bar{x}_2 < \underline{x}_1$. Furthermore, by using (3.4.29) and the equation (3.4.32) satisfied by y, we may easily check that A and B are positive constants.

Notice that by construction, w_1 (resp. w_2) is C^2 on $(0, \underline{x}_1) \cup (\underline{x}_1, \infty)$ (resp. $(0, \bar{x}_2) \cup (\bar{x}_2, \infty)$) and C^1 at $\underline{x}_1$ (resp. $\bar{x}_2$).

$\star$ By using Lemma 3.3.4, we now show that w_i, $i = 1, 2$, is a viscosity solution to the system :

$$\min\{rw_i - \mathcal{L}_i w_i - f_i, w_i - (w_j - g_{ij})\} \;=\; 0, \quad \text{on } (0,\infty),\ i,j = 1,2,\ j \neq i. \tag{3.4.33}$$

Since the proof is similar for both w_i, $i = 1, 2$, we only prove the result for w_1. We first check that

$$w_1 \geq w_2 - g_{12}, \quad \forall\, 0 < x < \underline{x}_1. \tag{3.4.34}$$

From the definition of w_1 and w_2 and using the fact that $g_{12} + g_{21} > 0$, it is straightforward to see that

$$w_1 \geq w_2 - g_{12}, \quad \forall\, 0 < x \leq \bar{x}_2. \tag{3.4.35}$$

Now, we need to prove that

$$G(x) := Ax^{m_1^+} + \hat{V}_1(x) - Bx^{m_2^-} - \hat{V}_2(x) + g_{12} \geq 0, \quad \forall\, \bar{x}_2 < x < \underline{x}_1. \tag{3.4.36}$$

We have $G(\bar{x}_2) = g_{12} + g_{21} > 0$ and $G(\underline{x}_1) = 0$. Suppose that there exists some $x_0 \in (\bar{x}_2, \underline{x}_1)$ such that $G(x_0) = 0$. We then deduce that there exists $x_3 \in (\bar{x}_0, \underline{x}_1)$ such that $G'(x_3) = 0$. As such, the equation $G'(x) = 0$ admits at least three solutions in $[\bar{x}_2, \underline{x}_1]$: $\{\bar{x}_2, x_3, \underline{x}_1\}$. However, a straightforward study of the function G shows that G' can take the value zero at most at two points in $(0,\infty)$. This leads to a contradiction, proving therefore (3.4.36).

By definition of w_1, we have in the classical sense

$$rw_1 - \mathcal{L}_1 w_1 - f = 0, \quad \text{on } (0, \underline{x}_1). \tag{3.4.37}$$

We now check that

$$rw_1 - \mathcal{L}_1 w_1 - f \geq 0, \quad \text{on } (\underline{x}_1, \infty) \tag{3.4.38}$$

holds true in the classical sense, and so a fortiori in the viscosity sense. By definition of w_1 on (x_1, ∞), and K_1, we have for all $x > \underline{x}_1$,

$$H(x) := rw_1(x) - \mathcal{L}_1 w_1(x) - f(x) = \frac{K_2 - K_1}{K_1} x^\gamma + m_2^- LBx^{m_2^-} - rg_{12},\ \forall x > \underline{x}_1, \tag{3.4.39}$$

where $L = \frac{1}{2}(\sigma_2^2 - \sigma_1^2)(m_2^- - 1) + b_2 - b_1$.

We distinguish two cases :

- First, if $L \geq 0$, the function H would be non-decreasing on $(0,\infty)$ with $\lim_{x\to 0^+} H(x) = -\infty$ and $\lim_{x\to\infty} H(x) = +\infty$. As such, it suffices to show that $H(\underline{x}_1) \geq 0$. From (3.4.24)-(3.4.25), we have

$$H(\underline{x}_1) \;=\; (K_2 - K_1)\left[\frac{m_1^+ - m_2^-}{K_1} - (m_1^+ - \gamma)m_2^- L\right] - rg_{12} + m_1^+ m_2^- g_{12} L.$$

Using relations (3.4.21), (3.4.24), (3.4.25), (3.4.29) and the definition of m_1^+ and m_2^-, we then obtain

$$H(\underline{x}_1) \;=\; \frac{m_1^+(m_1^+ - m_2^-)}{K_1(m_1^+ - \gamma)} - r \geq \frac{m_1^+}{K_1(m_1^+ - \gamma)} - r \geq 0.$$

- Second, if $L < 0$, it suffices to show that

$$\frac{K_2 - K_1}{K_1} x^\gamma - rg_{12} \geq 0, \quad \forall\, x > \underline{x}_1,$$

which is rather straightforward from (3.4.21) and (3.4.29) .
Relations (3.4.34), (3.4.37) (3.4.38) and the regularity of w_i, $i = 1, 2$, as constructed, mean that conditions of Lemma 3.3.4 are satisfied and we thus get the required assertion (3.4.33).

$\star$ Since $w_1(0^+) = 0 = (-g_{12})_+$, $w_2(0^+) = -g_{21} = (-g_{21})_+$, and w_1, $\hat{V}_2$ satisfy a linear growth condition, we deduce from (3.4.33) and uniqueness to the PDE system (3.4.1)-(3.4.2), that

$$v_1 \;=\; w_1, \qquad v_2 \;=\; w_2, \quad \text{ on } \ (0,\infty).$$

This proves $\underline{x}_1^* = \underline{x}_1$, $\mathcal{S}_1 = [\underline{x}_1, \infty)$ and $\bar{x}_2^* = \bar{x}_2$, $\mathcal{S}_2 = (0, \bar{x}_2]$.

3.4.2 Identical diffusion operators with different profit functions

In this subsection, we suppose that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, i.e. $b_1 = b_2 = b$, $\sigma_1 = \sigma_2 = \sigma > 0$. We then set $m^+ = m_1^+ = m_2^+$, $m^- = m_1^- = m_2^-$, and $\hat{X}^x = \hat{X}^{x,1} = \hat{X}^{x,2}$. Notice that in this case, the set Q_{ij}, i, $j = 1, 2$, $i \neq j$, introduced in Lemma 3.3.3, satisfies :

$$\begin{aligned} Q_{ij} \;&=\; \{x \in \mathcal{C}_j \ : (f_j - f_i)(x) - rg_{ij} \geq 0\} \\ &\subset \;\hat{Q}_{ij} := \{x > 0 \ : (f_j - f_i)(x) - rg_{ij} \geq 0\}. \end{aligned} \tag{3.4.40}$$

Once we are given the profit functions f_i, f_j, the set $\hat{Q}_{ij}$ can be explicitly computed. Moreover, we prove in the next key Lemma that the structure of $\hat{Q}_{ij}$, when it is connected, determines the same structure for the switching region $\mathcal{S}_i$.

Lemma 3.4.1 *Let $i, j = 1, 2$, $i \neq j$.*
1) Assume that

$$\sup_{x>0}(\hat{V}_j - \hat{V}_i)(x) \;>\; g_{ij}. \tag{3.4.41}$$

• *If there exists* $0 < \underline{x}_{ij} < \infty$ *such that*

$$\hat{Q}_{ij} = [\underline{x}_{ij}, \infty), \tag{3.4.42}$$

then $0 < \underline{x}_i^* < \infty$ *and*

$$\mathcal{S}_i = [\underline{x}_i^*, \infty).$$

• *If* $g_{ij} \leq 0$ *and there exists* $0 < \bar{x}_{ij} < \infty$ *such that*

$$\hat{Q}_{ij} = (0, \bar{x}_{ij}], \tag{3.4.43}$$

then $0 < \bar{x}_i^* < \infty$ *and*

$$\mathcal{S}_i = (0, \bar{x}_i^*].$$

2) If there exist $0 < \underline{x}_{ij} < \bar{x}_{ij} < \infty$ *such that*

$$\hat{Q}_{ij} = [\underline{x}_{ij}, \bar{x}_{ij}]. \tag{3.4.44}$$

Then $0 < \underline{x}_i^* < \bar{x}_i^* < \infty$ *and*

$$\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*].$$

3) If $g_{ij} \leq 0$ *and* $\hat{Q}_{ij} = (0, \infty)$, *then* $\mathcal{S}_i = (0, \infty)$ *and* $\mathcal{S}_j = \emptyset$.

Proof. 1) • Consider the case of condition (3.4.42). Since $\mathcal{S}_i \subset \hat{Q}_{ij}$ by Lemma 3.3.3, this implies $\underline{x}_i^* := \inf \mathcal{S}_i \geq \underline{x}_{ij} > 0$. We now claim that $\underline{x}_i^* < \infty$. On the contrary, the switching region $\mathcal{S}_i$ would be empty, and so v_i would satisfy on $(0, \infty)$:

$$r v_i - \mathcal{L} v_i - f_i = 0, \quad \text{on } (0, \infty).$$

Then, v_i would be on the form :

$$v_i(x) = A x^{m^+} + B x^{m^-} + \hat{V}_i(x), \quad x > 0.$$

Since $0 \leq v_i(0^+) < \infty$ and v_i is a nonnegative function satisfying a linear growth condition, and using the fact that $m^- < 0$ and $m^+ > 1$, we deduce that v_i should be equal to $\hat{V}_i$. Now, since we have $v_i \geq v_j - g_{ij} \geq \hat{V}_j - g_{ij}$, this would imply :

$$\hat{V}_j(x) - \hat{V}_i(x) \leq g_{ij}, \quad \forall x > 0.$$

This contradicts condition (3.4.41) and so $0 < \underline{x}_i^* < \infty$.

By definition of $\underline{x}_i^*$, we already know that $(0, \underline{x}_i^*) \subset \mathcal{C}_i$. We prove actually the equality, i.e. $\mathcal{S}_i = [\underline{x}_i^*, \infty)$ or $v_i(x) = v_j(x) - g_{ij}$ for all $x \geq \underline{x}_i^*$. Consider the function

$$w_i(x) = \begin{cases} v_i(x), & 0 < x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases}$$

We now check that w_i is a viscosity solution of

$$\min\{rw_i - \mathcal{L}w_i - f_i\ ,\ w_i - (v_j - g_{ij})\} \;=\; 0 \quad \text{on} \quad (0,\infty). \tag{3.4.45}$$

From Theorem 3.3.2, the function w_i is C^1 on $(0,\infty)$ and in particular at $\underline{x}_i^*$ where $w_i'(\underline{x}_i^*) = v_i'(\underline{x}_i^*) = v_j'(\underline{x}_i^*)$. We also know that $w_i = v_i$ is C^2 on $(0,\underline{x}_i^*) \subset \mathcal{C}_i$, and satisfies $rw_i - \mathcal{L}w_i - f_i = 0$, $w_i \geq (v_j - g_{ij})$ on $(0,\underline{x}_i^*)$. Hence, from Lemma 3.3.4, we only need to check the viscosity supersolution property of w_i to :

$$rw_i - \mathcal{L}w_i - f_i \;\geq\; 0, \quad \text{on} \;\; (\underline{x}_i^*,\infty). \tag{3.4.46}$$

For this, take some point $\bar{x} > \underline{x}_i^*$ and some smooth test function φ s.t. $\bar{x}$ is a local minimum of $w_i - \varphi$. Then, $\bar{x}$ is a local minimum of $v_j - (\varphi + g_{ij})$, and by the viscosity solution property of v_j to its Bellman PDE, we have

$$rv_j(\bar{x}) - \mathcal{L}\varphi(\bar{x}) - f_j(\bar{x}) \;\geq\; 0.$$

Now, since $\underline{x}_i^* \geq \underline{x}_{ij}$, we have $\bar{x} > \underline{x}_{ij}$ and so by (3.4.42), $\bar{x} \in \hat{Q}_{ij}$. Hence,

$$(f_j - f_i)(\bar{x}) - rg_{ij} \;\geq\; 0.$$

By adding the two previous inequalities, we also obtain the required supersolution inequality :

$$rw_i(\bar{x}) - \mathcal{L}\varphi(\bar{x}) - f_i(\bar{x}) \;\geq\; 0,$$

and so (3.4.45) is proved.

Since $w_i(0^+) = v_i(0^+)$ and w_i satisfies a linear growth condition, and from uniqueness of viscosity solution to PDE (3.4.45), we deduce that w_i is equal to v_i. In particular, we have $v_i(x) = v_j(x) - g_{ij}$ for $x \geq \underline{x}_i^*$, which shows that $\mathcal{S}_i = [\underline{x}_i^*,\infty)$.

• The case of condition (3.4.43) is dealt by same arguments as above : we first observe that $0 < \bar{x}_i^* := \sup \mathcal{S}_i < \infty$ under (3.4.41), and then show with Lemma 3.3.4 that the function

$$w_i(x) \;=\; \begin{cases} v_j(x) - g_{ij}, & 0 < x < \bar{x}_i^* \\ v_i(x), & x \geq \bar{x}_i^* \end{cases}$$

is a viscosity solution to

$$\min\{rw_i - \mathcal{L}w_i - f_i\ ,\ w_i - (v_j - g_{ij})\} \;=\; 0 \quad \text{on} \quad (0,\infty).$$

Then, under the condition that $g_{ij} \leq 0$, we see that $g_{ji} > 0$ by (3.2.7), and so $v_i(0^+) = -g_{ij} = (-g_{ji})_+ - g_{ij} = v_j(0^+) - g_{ij} = w_i(0^+)$. From uniqueness of viscosity solution to PDE (3.4.45), we conclude that $v_i = w_i$, and so $\mathcal{S}_i = (0, \bar{x}_i^*]$.

2) By Lemma 3.3.3 and (3.4.40), the condition (3.4.44) implies $0 < \underline{x}_{ij} \leq \underline{x}_i^* \leq \bar{x}_i^* \leq \bar{x}_{ij} < \infty$. We claim that $\underline{x}_i^* < \bar{x}_i^*$. Otherwise, $\mathcal{S}_i = \{\bar{x}_i^*\}$ and v_i would satisfy $rv_i - \mathcal{L}v_i - f_i =$

0 on $(0, \bar{x}_i^*) \cup (\bar{x}_i^*, \infty)$. By continuity and smooth-fit condition of v_i at $\bar{x}_i^*$, this implies that v_i satisfies actually

$$rv_i - \mathcal{L}v_i - f_i \quad = \quad 0, \quad x \in (0, \infty),$$

and so is in the form :

$$v_i(x) \quad = \quad Ax^{m^+} + Bx^{m^-} + \hat{V}_i(x), \quad x \in (0, \infty)$$

Since $0 \leq v_i(0^+) < \infty$ and v_i is nonnegative function satisfying a linear growth condition, this implies $A = B = 0$. Therefore, v_i is equal to $\hat{V}_i$, which also means that $\mathcal{S}_i = \emptyset$, a contradiction.

We now prove that $\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*]$. Let us consider the function

$$w_i(x) \quad = \quad \begin{cases} v_i(x), & x \in (0, \underline{x}_i^*) \cup (\bar{x}_i^*, \infty) \\ v_j(x) - g_{ij}, & x \in [\underline{x}_i^*, \bar{x}_i^*], \end{cases}$$

which is C^1 on $(0, \infty)$ and in particular on $\underline{x}_i^*$ and $\bar{x}_i^*$ from Theorem 3.3.2. Hence, by similar arguments as in case 1), using Lemma 3.3.4, we then show that w_i is a viscosity solution of

$$\min \{ rw_i - \mathcal{L}w_i - f_i \ , \ w_i - (v_j - g_{ij}) \} \quad = \quad 0. \tag{3.4.47}$$

Since $w_i(0^+) = v_i(0^+)$ and w_i satisfies a linear growth condition, and from uniqueness of viscosity solution to PDE (3.4.47), we deduce that w_i is equal to v_i. In particular, we have $v_i(x) = v_j(x) - g_{ij}$ for $x \in [\underline{x}_i^*, \bar{x}_i^*]$, which shows that $\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*]$.

3) Suppose that $g_{ij} \leq 0$ and $\hat{Q}_{ij} = (0, \infty)$. We shall prove that $\mathcal{S}_i = (0, \infty)$ and $\mathcal{S}_j = \emptyset$. To this end, we consider the smooth functions $w_i = \hat{V}_j - g_{ij}$ and $w_j = \hat{V}_j$. Then, recalling the ode satisfied by $\hat{V}_j$, and inequality (3.2.7), we get :

$$rw_j - \mathcal{L}w_j - f_j \ = \ 0, \qquad w_j - (w_i - g_{ji}) \ = \ g_{ij} + g_{ji} \ \geq \ 0.$$

Therefore w_j is a smooth (and so a viscosity) solution to :

$$\min [rw_j - \mathcal{L}w_j - f_j, w_j - (w_i - g_{ji})] \quad = \quad 0 \qquad \text{on} \ \ (0, \infty).$$

On the other hand, by definition of $\hat{Q}_{ij}$, which is supposed equal to $(0, \infty)$, we have :

$$\begin{aligned} rw_i(x) - \mathcal{L}w_i(x) - f_i(x) \quad &= \quad r\hat{V}_j(x) - \mathcal{L}\hat{V}_j(x) - f_j(x) + f_j(x) - f_i(x) - rg_{ij} \\ &= \quad f_j(x) - f_i(x) - rg_{ij} \ \geq \ 0, \quad \forall x > 0. \end{aligned}$$

Moreover, by construction we have $w_i = w_j - g_{ij}$. Therefore w_i is a smooth (and so a viscosity) solution to :

$$\min [rw_i - \mathcal{L}w_i - f_i, w_i - (w_j - g_{ij})] \quad = \quad 0 \qquad \text{on} \ \ (0, \infty).$$

Notice also that $g_{ji} > 0$ by (3.2.7) and since $g_{ij} \leq 0$. Hence, $w_i(0^+) = -g_{ij} = (-g_{ij})_+ = v_i(0^+)$, $w_j(0^+) = 0 = (-g_{ji})_+ = v_j(0^+)$. From uniqueness result of Theorem 3.3.1, we deduce that $v_i = w_i$, $v_j = w_j$, which proves that $\mathcal{S}_i = (0, \infty)$, $\mathcal{S}_j = \emptyset$. $\square$

We shall now provide explicit solutions to the switching problem under general assumptions on the running profit functions, which include several interesting cases for applications :

(HF) There exists $\hat{x} \in \mathbb{R}_+$ s.t the function $F := f_2 - f_1$ is decreasing on $(0, \hat{x})$, increasing on $[\hat{x}, \infty)$, and $F(\infty) := \lim_{x \to \infty} F(x) > 0$, $\quad g_{12} > 0$.

Under **(HF)**, there exists some $\bar{x} \in \mathbb{R}_+$ ($\bar{x} > \hat{x}$ if $\hat{x} > 0$ and $\bar{x} = 0$ if $\hat{x} = 0$) from which F is positive : $F(x) > 0$ for $x > \bar{x}$. Economically speaking, condition **(HF)** means that the profit in regime 2 is "better" than profit in regime 1 from a certain level $\bar{x}$, eventually equal to zero, and the improvement becomes then better and better. Moreover, since profit in regime 2 is better than the one in regime 1, it is natural to assume that the corresponding switching cost g_{12} from regime 1 to 2 should be positive. However, we shall consider both cases where g_{21} is positive and non-positive. Notice that $F(\hat{x}) < 0$ if $\hat{x} > 0$, $F(\hat{x}) = 0$ if $\hat{x} = 0$, and we do not assume necessarily $F(\infty) = \infty$.

Example 3.4.1 A typical example of different running profit functions satisfying **(HF)** is given by

$$f_i(x) = k_i x^{\gamma_i}, \quad i = 1, 2, \quad \text{with } 0 < \gamma_1 < \gamma_2 < 1, \quad k_1 \in \mathbb{R}_+, \; k_2 > 0. \tag{3.4.48}$$

In this case, $\hat{x} = (\frac{k_1 \gamma_1}{k_2 \gamma_2})^{\frac{1}{\gamma_2 - \gamma_1}}$, and $\lim_{x \to \infty} F(x) = \infty$.

Another example of profit functions of interest in applications is the case where the profit function in regime 1 is $f_1 = 0$, and the other f_2 is increasing. In this case, assumption **(HF)** is satisfied with $\hat{x} = 0$.

The next proposition states the form of the switching regions in regimes 1 and 2, depending on the parameter values.

Proposition 3.4.1 *Assume that* **(HF)** *holds.*
1) (i) If $rg_{12} \geq F(\infty)$, *then* $\underline{x}_1^* = \infty$, *i.e.* $\mathcal{S}_1 = \emptyset$.
(ii) If $rg_{12} < F(\infty)$, *then* $\underline{x}_1^* \in (0, \infty)$ *and* $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$.

2) (i) If $rg_{21} \geq -F(\hat{x})$, *then* $\mathcal{S}_2 = \emptyset$.
(ii) If $0 < rg_{21} < -F(\hat{x})$, *then* $0 < \underline{x}_2^* < \bar{x}_2^* < \underline{x}_1^*$, *and* $\mathcal{S}_2 = [\underline{x}_2^*, \bar{x}_2^*]$.
(iii) If $g_{21} \leq 0$ *and* $-F(\infty) < rg_{21} < -F(\hat{x})$, *then* $0 = \underline{x}_2^* < \bar{x}_2^* < \underline{x}_1^*$, *and* $\mathcal{S}_2 = (0, \bar{x}_2^*]$.
(iv) If $rg_{21} \leq -F(\infty)$, *then* $\mathcal{S}_2 = (0, \infty)$.

Proof. 1) From Lemma 3.3.3, we have

$$\hat{Q}_{12} \quad = \quad \{x > 0 \ : F(x) \geq rg_{12}\}. \tag{3.4.49}$$

Since $g_{12} > 0$, and $f_i(0) = 0$, we have $F(0) = 0 < rg_{12}$. Under **(HF)**, we then distinguish the two following cases :
(i) If $rg_{12} \geq F(\infty)$, then $\hat{Q}_{12} = \emptyset$, and so by Lemma 3.3.3 and (3.4.40), $\mathcal{S}_1 = \emptyset$.
(ii) If $rg_{12} < F(\infty)$, then there exists $\hat{x}_{12} \in (0, \infty)$ such that

$$\hat{Q}_{12} \quad = \quad [\underline{x}_{12}, \infty). \tag{3.4.50}$$

Moreover, since

$$(\hat{V}_2 - \hat{V}_1)(x) \quad = \quad E\left[\int_0^\infty e^{-rt} F(\hat{X}_t^x) dt\right], \quad \forall x > 0,$$

and F is lower-bounded, we obtain by Fatou's lemma :

$$\liminf_{x \to \infty} (\hat{V}_2 - \hat{V}_1)(x) \quad \geq \quad E\left[\int_0^\infty e^{-rt} F(\infty) dt\right] \; = \; \frac{F(\infty)}{r} \; > \; g_{12}.$$

Hence, conditions (3.4.41)-(3.4.42) with $i = 1$, $j = 2$, are satisfied, and we obtain the first assertion by Lemma 3.4.1 1).
2) From Lemma 3.3.3, we have

$$\hat{Q}_{21} \quad = \quad \{x > 0 \ : -F(x) \geq rg_{21}\}. \tag{3.4.51}$$

Under **(HF)**, we distinguish the following cases :

► (i1) If $rg_{21} > -F(\hat{x})$, then $\hat{Q}_{21} = \emptyset$, and so $\mathcal{S}_2 = \emptyset$.
(i2) If $rg_{21} = -F(\hat{x})$, then either $\hat{x} = 0$ and so $\mathcal{S}_2 = \hat{Q}_{21} = \emptyset$, or $\hat{x} > 0$, and so $\hat{Q}_{21} = \{\hat{x}\}$, $\mathcal{S}_2 \subset \{\hat{x}\}$. In this last case, v_2 satisfies $rv_2 - \mathcal{L}v_2 - f_2 = 0$ on $(0, \hat{x}) \cup (\hat{x}, \infty)$. By continuity and smooth-fit condition of v_2 at $\hat{x}$, this implies that v_2 satisfies actually

$$rv_2 - \mathcal{L}v_2 - f_2 \quad = \quad 0, \quad x \in (0, \infty),$$

and so is in the form :

$$v_2(x) \quad = \quad Ax^{m^+} + Bx^{m^-} + \hat{V}_2(x), \quad x \in (0, \infty)$$

Recalling that $0 \leq v_2(0^+) < \infty$ and v_2 is a nonnegative function satisfying a linear growth condition, this implies $A = B = 0$. Therefore, v_2 is equal to $\hat{V}_2$, which also means that $\mathcal{S}_2$ $= \emptyset$.
► If $rg_{21} < -F(\hat{x})$, we need to distinguish three subcases depending on g_{21} :

- If $g_{21} > 0$, then there exist $0 < \underline{x}_{21} < \hat{x} < \bar{x}_{21} < \infty$ such that

$$\hat{Q}_{21} \quad = \quad [\underline{x}_{21}, \bar{x}_{21}]. \tag{3.4.52}$$

We then conclude with Lemma 3.4.1 2) for $i = 2$, $j = 1$.

- If $g_{21} \leq 0$ with $rg_{21} > -F(\infty)$, then there exists $\bar{x}_{21} < \infty$ s.t.

$$\hat{Q}_{21} = (0, \bar{x}_{21}].$$

  Moreover, we clearly have $\sup_{x>0}(\hat{V}_1 - \hat{V}_2)(x) > (\hat{V}_1 - \hat{V}_2)(0) = 0 \geq g_{21}$. Hence, conditions (3.4.41) and (3.4.43) with $i = 2$, $j = 1$ are satisfied, and we deduce from Lemma 3.4.1 1) that $\mathcal{S}_2 = (0, \bar{x}_2^*]$ with $0 < \bar{x}_2^* < \infty$.

- If $rg_{21} \leq -F(\infty)$, then $\hat{Q}_{21} = (0, \infty)$, and we deduce from Lemma 3.4.1 3) for $i = 2$, $j = 1$, that $\mathcal{S}_2 = (0, \infty)$.

Finally, in the two above subcases when $\mathcal{S}_2 = [\underline{x}_2^*, \bar{x}_2^*]$ or $(0, \bar{x}_2^*]$, we notice that $\bar{x}_2^* < \underline{x}_1^*$ since $\mathcal{S}_2 \subset \mathcal{C}_1 = (0, \infty) \setminus \mathcal{S}_1$, which is equal, from 1), either to $(0, \infty)$ when $\underline{x}_1^* = \infty$ or to $(0, \underline{x}_1^*)$. □

Remark 3.4.2 In our viscosity solutions approach, the structure of the switching regions is derived from the smooth fit property of the value functions, uniqueness result for viscosity solutions and Lemma 3.3.3. This contrasts with the classical verification approach where the structure of switching regions should be guessed ad-hoc and checked a posteriori by a verification argument.

Economic interpretation.

The previous proposition shows that, under **(HF)**, the switching region in regime 1 has two forms depending on the size of its corresponding positive switching cost : If g_{12} is larger than the "maximum net" profit $F(\infty)$ that one can expect by changing of regime (case 1) (i), which may occur only if $F(\infty) < \infty$), then one has no interest to switch of regime, and one always stay in regime 1, i.e. $\mathcal{C}_1 = (0, \infty)$. However, if this switching cost is smaller than $F(\infty)$ (case 1) (ii), which always holds true when $F(\infty) = \infty$), then there is some positive threshold from which it is optimal to change of regime.

The structure of the switching region in regime 2 exhibits several different forms depending on the sign and size of its corresponding switching cost g_{21} with respect to the values $-F(\infty) < 0$ and $-F(\hat{x}) \geq 0$. If g_{21} is nonnegative larger than $-F(\hat{x})$ (case 2) (i)), then one has no interest to switch of regime, and one always stay in regime 2, i.e. $\mathcal{C}_2 = (0, \infty)$. If g_{21} is positive, but not too large (case 2) (ii)), then there exists some bounded closed interval, which is not a neighborhood of zero, where it is optimal to change of regime. Finally, when the switching cost g_{21} is negative, it is optimal to switch to regime 1 at least for small values of the state. Actually, if the negative cost g_{21} is larger than $-F(\infty)$ (case 2) (iii), which always holds true for negative cost when $F(\infty) = \infty$), then the switching region is a bounded neighborhood of 0. Moreover, if the cost is negative large enough (case 2) (iv), which may occur only if $F(\infty) < \infty$), then it is optimal to change of regime for every values of the state.

By combining the different cases for regimes 1 and 2, and observing that case 2) (iv) is not compatible with case 1) (ii) by (3.2.7), we then have a priori seven different forms for both switching regions. These forms reduce actually to three when $F(\infty) = \infty$. The various structures of the switching regions are depicted in Figure II.

Finally, we complete results of Proposition 3.4.1 by providing the explicit solutions for the value functions and the corresponding boundaries of the switching regions in the seven different cases depending on the model parameter values.

Theorem 3.4.2 *Assume that* **(HF)** *holds.*
1) *If* $rg_{12} < F(\infty)$ *and* $rg_{21} \geq -F(\hat{x})$, *then*

$$
\begin{aligned}
v_1(x) &= \begin{cases} Ax^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases} \\
v_2(x) &= \hat{V}_2(x),
\end{aligned}
$$

where the constants A *and* $\underline{x}_1^*$ *are determined by the continuity and smooth-fit conditions of* v_1 *at* $\underline{x}_1^*$ *:*

$$
\begin{aligned}
A(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) &= \hat{V}_2(\underline{x}_1^*) - g_{12} \\
Am^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) &= \hat{V}_2'(\underline{x}_1^*).
\end{aligned}
$$

In regime 1, *it is optimal to switch to regime* 2 *whenever the state process* X *exceeds the threshold* $\underline{x}_1^*$, *while when we are in regime* 2, *it is optimal never to switch.*
2) *If* $rg_{12} < F(\infty)$ *and* $0 < rg_{21} < -F(\hat{x})$, *then*

$$
v_1(x) = \begin{cases} A_1x^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases} \tag{3.4.53}
$$

$$
v_2(x) = \begin{cases} A_2x^{m^+} + \hat{V}_2(x), & x < \underline{x}_2^* \\ v_1(x) - g_{21}, & \underline{x}_2^* \leq x \leq \bar{x}_2^* \\ B_2x^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases} \tag{3.4.54}
$$

where the constants A_1 *and* $\underline{x}_1^*$ *are determined by the continuity and smooth-fit conditions of* v_1 *at* $\underline{x}_1^*$, *and the constants* A_2, B_2, $\underline{x}_2^*$, $\bar{x}_2^*$ *are determined by the continuity and smooth-fit conditions of* v_2 *at* $\underline{x}_2^*$ *and* $\bar{x}_2^*$ *:*

$$
\begin{aligned}
A_1(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) &= B_2(\underline{x}_1^*)^{m^-} + \hat{V}_2(\underline{x}_1^*) - g_{12} && (3.4.55) \\
A_1m^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) &= B_2m^-(\underline{x}_1^*)^{m^--1} + \hat{V}_2'(\underline{x}_1^*) && (3.4.56) \\
A_2(\underline{x}_2^*)^{m^+} + \hat{V}_2(\underline{x}_2^*) &= A_1(\underline{x}_2^*)^{m^+} + \hat{V}_1(\underline{x}_2^*) - g_{21} && (3.4.57) \\
A_2m^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_2'(\underline{x}_2^*) &= A_1m^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_1'(\underline{x}_2^*) && (3.4.58) \\
A_1(\bar{x}_2^*)^{m^+} + \hat{V}_1(\bar{x}_2^*) - g_{21} &= B_2(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) && (3.4.59) \\
A_1m^+(\bar{x}_2^*)^{m^+-1} + \hat{V}_1'(\bar{x}_2^*) &= B_2m^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*). && (3.4.60)
\end{aligned}
$$

Figure II

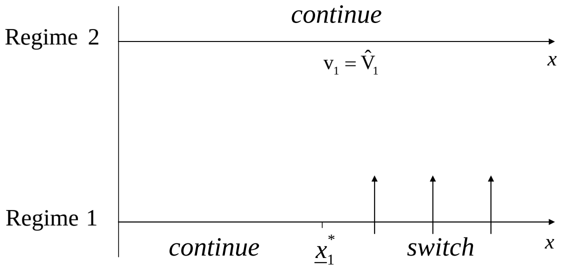


Figure II.1: $rg_{12} < F(\infty)$, $rg_{21} \geq -F(\hat{x})$

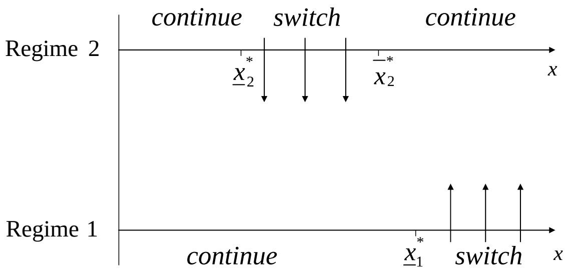


Figure II.2: $rg_{12} < F(\infty)$, $0 < rg_{21} < -F(\hat{x})$

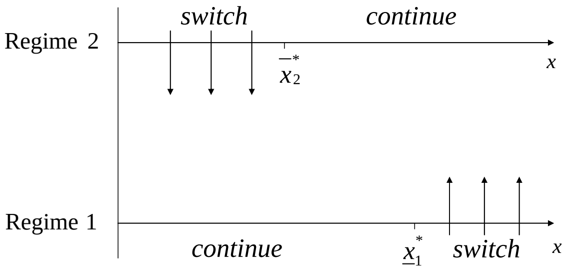


Figure II.3: $rg_{12} < F(\infty)$, $g_{21} \leq 0$, $-F(\infty) < rg_{21} < -F(\hat{x})$

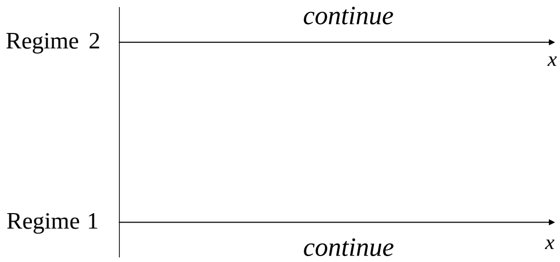


Figure II.4: $rg_{12} \geq F(\infty)$, $rg_{21} > -F(\hat{x})$

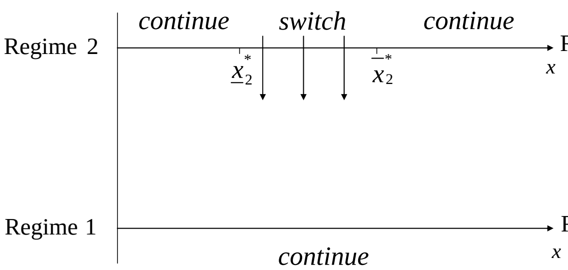


Figure II.5: $rg_{12} \geq F(\infty)$, $0 < rg_{21} < -F(\hat{x})$

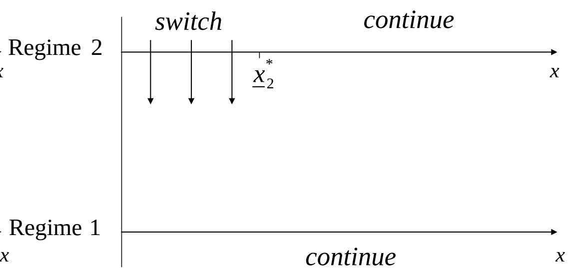


Figure II.6: $rg_{12} \geq F(\infty)$, $g_{21} \leq 0$, $F(\infty) < rg_{21} < -F(\hat{x})$

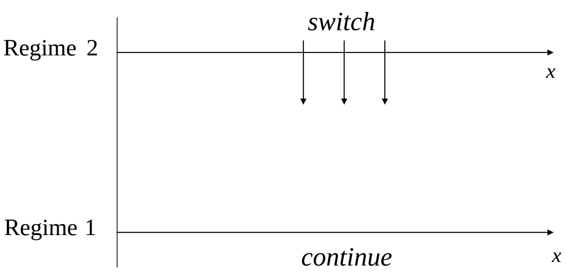


Figure II.7: $rg_{12} \geq F(\infty)$, $g_{21} \leq -F(\infty)$

In regime 1, it is optimal to switch to regime 2 whenever the state process X exceeds the threshold $\underline{x}_1^*$, while when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies between $\underline{x}_2^*$ and $\bar{x}_2^*$.

3) *If $rg_{12} < F(\infty)$ and $g_{21} \leq 0$ with $-F(\infty) < rg_{21} < -F(\hat{x})$, then*

$$v_1(x) = \begin{cases} Ax^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases}$$

$$v_2(x) = \begin{cases} v_1(x) - g_{21}, & 0 < x \leq \bar{x}_2^* \\ Bx^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases}$$

where the constants A and $\underline{x}_1^*$ are determined by the continuity and smooth-fit conditions of v_1 at $\underline{x}_1^*$, and the constants B and $\bar{x}_2^*$ are determined by the continuity and smooth-fit conditions of v_2 at $\bar{x}_2^*$:

$$\begin{aligned} A(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) &= B(\underline{x}_1^*)^{m^-} + \hat{V}_2(\underline{x}_1^*) - g_{12} \\ Am^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) &= Bm^-(\underline{x}_1^*)^{m^--1} + \hat{V}_2'(\underline{x}_1^*) \\ A(\bar{x}_2^*)^{m^+} + \hat{V}_1(\bar{x}_2^*) - g_{21} &= B(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) \\ Am^+(\bar{x}_2^*)^{m^+-1} + \hat{V}_1'(\bar{x}_2^*) &= Bm^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*). \end{aligned}$$

4) *If $rg_{12} \geq F(\infty)$ and $rg_{21} \geq -F(\hat{x})$, then $v_1 = \hat{V}_1$, $v_2 = \hat{V}_2$. It is optimal never to switch in both regimes 1 and 2.*

5) *If $rg_{12} \geq F(\infty)$ and $0 < rg_{21} < -F(\hat{x})$, then*

$$\begin{aligned} v_1(x) &= \hat{V}_1(x) \\ v_2(x) &= \begin{cases} Ax^{m^+} + \hat{V}_2(x), & x < \underline{x}_2^* \\ v_1(x) - g_{21}, & \underline{x}_2^* \leq x \leq \bar{x}_2^* \\ Bx^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases} \end{aligned}$$

where the constants A, B, $\underline{x}_2^*$, $\bar{x}_2^*$ are determined by the continuity and smooth-fit conditions of v_2 at $\underline{x}_2^*$ and $\bar{x}_2^*$:

$$\begin{aligned} A(\underline{x}_2^*)^{m^+} + \hat{V}_2(\underline{x}_2^*) &= \hat{V}_1(\underline{x}_2^*) - g_{21} \\ Am^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_2'(\underline{x}_2^*) &= \hat{V}_1'(\underline{x}_2^*) \\ \hat{V}_1(\bar{x}_2^*) - g_{21} &= B(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) \\ \hat{V}_1'(\bar{x}_2^*) &= Bm^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*). \end{aligned}$$

In regime 1, it is optimal never to switch, while when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies between $\underline{x}_2^*$ and $\bar{x}_2^*$.

6) *If $rg_{12} \geq F(\infty)$ and $g_{21} \leq 0$ with $-F(\infty) < rg_{21} < -F(\hat{x})$, then*

$$\begin{aligned} v_1(x) &= \hat{V}_1(x) \\ v_2(x) &= \begin{cases} v_1(x) - g_{21}, & 0 < x \leq \bar{x}_2^* \\ Bx^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases} \end{aligned}$$

where the constants B and $\bar{x}_2^*$ are determined by the continuity and smooth-fit conditions of v_2 at $\bar{x}_2^*$:

$$\begin{aligned} \hat{V}_1(\bar{x}_2^*) - g_{21} &= B(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) \\ \hat{V}_1'(\bar{x}_2^*) &= Bm^-(\bar{x}_2^*)^{m^- - 1} + \hat{V}_2'(\bar{x}_2^*). \end{aligned}$$

In regime 1, it is optimal never to switch, while when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies below $\bar{x}_2^*$.

7) *If $rg_{12} \geq F(\infty)$ and $rg_{21} \leq -F(\infty)$, then $v_1 = \hat{V}_1$ and $v_2 = v_1 - g_{12}$. In regime 1, it is optimal never to switch, while when we are in regime 2, it is always optimal to switch to regime 1.*

Proof. We prove the result only for the case **2)** since the other cases are dealt similarly and are even simpler. This case **2)** corresponds to the combination of cases 1) (ii) and 2) (ii) in Proposition 3.4.1. We then have $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$, which means that $v_1 = v_2 - g_{12}$ on $[\underline{x}_1^*, \infty)$ and v_1 is solution to $rv_1 - \mathcal{L}v_1 - f_1 = 0$ on $(0, \underline{x}_1^*)$. Since $0 \leq v_1(0^+) < \infty$, v_1 should have the form expressed in (3.4.53). Moreover, $\mathcal{S}_2 = [\underline{x}_2^*, \bar{x}_2^*]$, which means that $v_2 = v_1 - g_{21}$ on $[\underline{x}_2^*, \bar{x}_2^*]$, and v_2 satisfies on $\mathcal{C}_2 = (0, \underline{x}_2^*) \cup (\bar{x}_2^*, \infty)$: $rv_2 - \mathcal{L}v_2 - f_2 = 0$. Recalling again that $0 \leq v_2(0^+) < \infty$ and v_2 satisfies a linear growth condition, we deduce that v_2 has the form expressed in (3.4.54). Finally, the constants A_1, $\underline{x}_1^*$, which characterize completely v_1, and the constants A_2, B_2, $\underline{x}_2^*$, $\bar{x}_2^*$, which characterize completely v_2, are determined by the six relations (3.4.55)-(3.4.56)-(3.4.57)-(3.4.58)-(3.4.59)-(3.4.60) resulting from the continuity and smooth-fit conditions of v_1 at $\underline{x}_1^*$ and v_2 at $\underline{x}_2^*$ and $\bar{x}_2^*$, and recalling that $\bar{x}_2^* < \underline{x}_1^*$. □

Remark 3.4.3 In the classical approach, for instance in the case **2)**, we construct a priori a candidate solution in the form (3.4.53)-(3.4.54), and we have to check the existence of a sextuple solution to (3.4.55)-(3.4.56)-(3.4.57)-(3.4.58)-(3.4.59)-(3.4.60), which may be somewhat tedious! Here, by the viscosity solutions approach, and since we already state the smooth-fit C^1 property of the value functions, we know a priori the existence of a sextuple solution to (3.4.55)-(3.4.56)-(3.4.57)-(3.4.58)-(3.4.59)-(3.4.60).

Appendix: Proof of comparison principle

In this section, we prove a comparison principle for the system of variational inequalities (3.3.8). The comparison result in [64] for switching problems in finite horizon does not apply in our context. Inspired by [41], we first produce some suitable perturbation of viscosity supersolution to deal with the switching obstacle, and then follow the general viscosity solution technique, see e.g. [19].

Theorem 3.A.1 *Suppose u_i, $i \in \mathbb{I}_d$, are continuous viscosity subsolutions to the system of variational inequalities* (3.3.8) *on* $(0, \infty)$, *and w_i, $i \in \mathbb{I}_d$, are continuous viscosity supersolutions to the system of variational inequalities* (3.3.8) *on* $(0, \infty)$, *satisfying the boundary* *conditions* $u_i(0^+) \leq w_i(0^+)$*,* $i \in I_d$*, and the linear growth condition :*

$$|u_i(x)| + |w_i(x)| \quad \leq \quad C_1 + C_2 x, \quad \forall x \in (0,\infty),\ i \in \mathbb{I}_d, \tag{3.A.1}$$

for some positive constants C_1 *and* C_2*. Then,*

$$u_i \quad \leq \quad w_i, \quad \textit{on } (0,\infty),\ \forall i \in \mathbb{I}_d.$$

Proof. Step 1. Let u_i and w_i, $i \in \mathbb{I}_d$, as in Theorem 3.A.1. We first construct strict supersolutions to the system (3.3.8) with suitable perturbations of w_i, $i \in \mathbb{I}_d$. We set

$$h(x) \quad = \quad C_1' + C_2' x^p, \quad x > 0,$$

where C_1', $C_2' > 0$ and $p > 1$ are positive constants to be determined later. We then define for all $\lambda \in (0,1)$, the continuous functions on $(0,\infty)$ by :

$$w_i^\lambda = (1-\lambda) w_i + \lambda(h + \alpha_i), \quad i \in \mathbb{I}_d,$$

where $\alpha_i = \min_{j\neq i} g_{ji}$. We then see that for all $\lambda \in (0,1)$, $i \in \mathbb{I}_d$:

$$\begin{aligned}
w_i^\lambda - \max_{j\neq i}(w_j^\lambda - g_{ij}) \quad &= \quad \lambda\alpha_i + (1-\lambda)w_i - \max_{j\neq i}[(1-\lambda)(w_j - g_{ij}) + \lambda\alpha_j - \lambda g_{ij}] \\
&\geq \quad (1-\lambda)[w_i - \max_{j\neq i}(w_j - g_{ij})] + \lambda\left(\alpha_i + \min_{j\neq i}(g_{ij} - \alpha_j)\right) \\
&\geq \quad \lambda \min_{i\in\mathbb{I}_d}\left(\alpha_i + \min_{j\neq i}(g_{ij} - \alpha_j)\right) \\
&\geq \quad \lambda\underline{\nu}
\end{aligned} \tag{3.A.2}$$

where $\underline{\nu} := \min_{i\in\mathbb{I}_d}\left[\alpha_i + \min_{j\neq i}(g_{ij} - \alpha_j)\right]$ is a constant independent of i. We now check that $\underline{\nu}$ > 0, i.e. $\nu_i := \alpha_i + \min_{j\neq i}(g_{ij} - \alpha_j) > 0$, $\forall i \in \mathbb{I}_d$. Indeed, fix $i \in \mathbb{I}_d$, and let $k \in \mathbb{I}_d$ such that $\min_{j\neq i}(g_{ij} - \alpha_j) = g_{ik} - \alpha_k$ and set $\underline{i}$ such that $\alpha_i = \min_{j\neq i} g_{ji} = g_{\underline{i}i}$. We then have

$$\nu_i \quad = \quad g_{\underline{i}i} + g_{ik} - \min_{j\neq k} g_{jk} \ > \ g_{\underline{i}k} - \min_{j\neq k} g_{jk} \ \geq \ 0,$$

by (3.2.6) and thus $\underline{\nu} > 0$.

By definition of the Fenchel Legendre in (3.2.5), and by setting $\tilde{f}(1) = \max_{i\in\mathbb{I}_d} \tilde{f}_i(1)$, we have for all $i \in \mathbb{I}_d$,

$$f_i(x) \quad \leq \quad \tilde{f}(1) + x \ \leq \tilde{f}(1) + 1 + x^p, \quad \forall x > 0.$$

Moreover, recalling that $r > b := \max_i b_i$, we can choose $p > 1$ s.t.

$$\rho \ := \ r - pb - \frac{1}{2}\sigma^2 p(p-1) \quad > \quad 0,$$

where we set $\sigma := \max_i \sigma_i > 0$. By choosing

$$C_1' \;\geq\; \frac{2+\tilde{f}(1)}{r} - \min_i \alpha_i, \qquad C_2' \;\geq\; \frac{1}{\rho},$$

we then have for all $i \in \mathbb{I}_d$,

$$\begin{aligned} rh(x) - \mathcal{L}_i h(x) - f_i(x) &= rC_1' + C_2' x^p [r - pb_i - \frac{1}{2}\sigma_i^2 p(p-1)] - f_i(x) \\ &\geq rC_1' + \rho C_2' x^p - f_i(x) \\ &\geq 1, \qquad \forall x > 0. \end{aligned} \tag{3.A.3}$$

From (3.A.2) and (3.A.3), we then deduce that for all $i \in \mathbb{I}_d$, $\lambda \in (0,1)$, w_i^λ is a supersolution to

$$\min\left\{ rw_i^\lambda - \mathcal{L}_i w_i^\lambda - f_i, w_i^\lambda - \max_{j \neq i}(w_j^\lambda - g_{ij}) \right\} \;\geq\; \lambda\delta, \quad \text{on } (0,\infty), \tag{3.A.4}$$

where $\delta = \underline{\nu} \wedge 1 > 0$.

<u>*Step 2.*</u> In order to prove the comparison principle, it suffices to show that for all $\lambda \in (0,1)$:

$$\max_{j \in \mathbb{I}_d} \sup_{(0,+\infty)} (u_j - w_j^\lambda) \;\leq\; 0$$

since the required result is obtained by letting λ to 0. We argue by contradiction and suppose that there exists some $\lambda \in (0,1)$ and $i \in \mathbb{I}_d$ s.t.

$$\theta := \max_{j \in \mathbb{I}_d} \sup_{(0,+\infty)} (u_j - w_j^\lambda) \;=\; \sup_{(0,+\infty)} (u_i - w_i^\lambda) \;>\; 0. \tag{3.A.5}$$

From the linear growth condition (3.A.1), and since $p > 1$, we observe that $u_i(x) - w_i^\lambda(x)$ goes to $-\infty$ when x goes to infinity. By choosing also $C_1' \geq \max_i w_i(0^+)$, we then have $u_i(0^+) - w_i^\lambda(0^+) = u_i(0^+) - w_i(0^+) + \lambda(w_i(0^+) - C_1') \leq 0$. Hence, by continuity of the functions u_i and w_i^λ, there exists $x_0 \in (0,\infty)$ s.t.

$$\theta \;=\; u_i(x_0) - w_i^\lambda(x_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned} \Phi_\varepsilon(x,y) &= u_i(x) - w_i^\lambda(y) - \phi_\varepsilon(x,y), \\ \phi_\varepsilon(x,y) &= \frac{1}{4}|x - x_0|^4 + \frac{1}{2\varepsilon}|x-y|^2, \end{aligned}$$

for all $x, y \in (0,\infty)$. By standard arguments in comparison principle, the function Φ_ε attains a maximum in $(x_\varepsilon, y_\varepsilon) \in (0,\infty)^2$, which converges (up to a subsequence) to (x_0,x_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \;=\; 0. \tag{3.A.6}$$

Applying Theorem 3.2 in [19], we get the existence of M_ε, $N_\varepsilon \in \mathbb{R}$ such that:

$$\begin{aligned}(p_\varepsilon, M_\varepsilon) &\in J^{2,+}u_i(x_\varepsilon),\\ (q_\varepsilon, N_\varepsilon) &\in J^{2,-}w_i^\lambda(y_\varepsilon)\end{aligned}$$

where

$$\begin{aligned}p_\varepsilon &= D_x\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\\ q_\varepsilon &= -D_y\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)\end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & -N_\varepsilon \end{pmatrix} \leq D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + \varepsilon\left(D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon)\right)^2 \tag{3.A.7}$$

with

$$D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \begin{pmatrix} 3(x_\varepsilon - x_0)^2 + \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix},$$

By writing the viscosity subsolution property (3.3.9) of u_i and the viscosity strict supersolution property (3.A.4) of w_i^λ, we have the following inequalities:

$$\min\left\{ru_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right)b_i x_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - f_i(x_\varepsilon),\right.$$
$$\left. u_i(x_\varepsilon) - \max_{j\neq i}(u_j - g_{ij})(x_\varepsilon)\right\} \leq 0 \tag{3.A.8}$$
$$\min\left\{rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)b_i y_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - f_i(y_\varepsilon),\right.$$
$$\left. w_i^\lambda(y_\varepsilon) - \max_{j\neq i}(w_j^\lambda - g_{ij})(y_\varepsilon)\right\} \geq \lambda\delta \tag{3.A.9}$$

We then distinguish the following two cases :

(1) $u_i(x_\varepsilon) - \max_{j\neq i}(u_j - g_{ij})(x_\varepsilon) \leq 0$ in (3.A.8).
By sending $\varepsilon \to 0$, this implies

$$u_i(x_0) - \max_{j\neq i}(u_j - g_{ij})(x_0) \leq 0. \tag{3.A.10}$$

On the other hand, we have by (3.A.9) :

$$w_i^\lambda(y_\varepsilon) - \max_{j\neq i}(w_j^\lambda - g_{ij})(y_\varepsilon) \geq \lambda\delta,$$

so that by sending ε to zero :

$$w_i^\lambda(x_0) - \max_{j\neq i}(w_j^\lambda - g_{ij})(x_0) \geq \lambda\delta. \tag{3.A.11}$$

Combining (3.A.10) and (3.A.11), we obtain :

$$\begin{aligned}\theta \ = \ u_i(x_0) - w_i^\lambda(x_0) &\leq -\lambda\delta + \max_{j\neq i}(u_j - g_{ij})(x_0) - \max_{j\neq i}(w_j^\lambda - g_{ij})(x_0) \\ &\leq -\lambda\delta + \max_{j\neq i}(u_j - w_j^\lambda)(x_0) \\ &\leq -\lambda\delta + \theta,\end{aligned}$$

which is a contradiction.

(2) $ru_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) b_i x_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - f_i(x_\varepsilon) \leq 0$ in (3.A.8).
Since by (3.A.9), we also have :

$$rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)b_i y_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - f_i(y_\varepsilon) \ \geq \ \lambda\delta,$$

this yields by combining the above two inequalities :

$$\begin{aligned}&ru_i(x_\varepsilon) - rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon}b_i(x_\varepsilon - y_\varepsilon)^2 - (x_\varepsilon - x_0)^3 b_i x_\varepsilon \\ &\quad + \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon + f_i(y_\varepsilon) - f_i(x_\varepsilon) \ \leq \ -\lambda\delta. \end{aligned} \tag{3.A.12}$$

Now, from (3.A.7), we have :

$$\frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon \leq \frac{3}{2\varepsilon}\sigma_i^2(x_\varepsilon - y_\varepsilon)^2 + \frac{3}{2}\sigma_i^2 x_\varepsilon^2 (x_\varepsilon - x_0)^2 \left(3\varepsilon(x_\varepsilon - x_0)^2 + 2\right),$$

so that by plugging into (3.A.12) :

$$\begin{aligned} r\left(u_i(x_\varepsilon) - w_i^\lambda(y_\varepsilon)\right) \ &\leq \ \frac{1}{\varepsilon}b_i(x_\varepsilon - y_\varepsilon)^2 + (x_\varepsilon - x_0)^3 b_i x_\varepsilon + \frac{3}{2\varepsilon}\sigma_i^2(x_\varepsilon - y_\varepsilon)^2 \\ &\quad + \frac{3}{2}\sigma_i^2 x_\varepsilon^2 (x_\varepsilon - x_0)^2 \left(3\varepsilon(x_\varepsilon - x_0)^2 + 2\right) + f_i(y_\varepsilon) - f_i(x_\varepsilon) - \lambda\delta \end{aligned}$$

By sending ε to zero, and using (3.A.6), continuity of f_i, we obtain the required contradiction: $r\theta \leq -\lambda\delta < 0$. This ends the proof of Theorem 3.A.1. $\square$

Chapter 4

A mixed singular/switching control problem for a dividend policy with reversible technology investment

Joint paper with Huyên PHAM and Stéphane VILLENEUVE, submitted to *Annals of Applied Probability.*

Abstract : We consider a mixed stochastic control problem that arises in Mathematical Finance literature with the study of interactions between dividend policy and investment. This problem combines features of both optimal switching and singular control. We prove that our mixed problem can be decoupled in two pure optimal stopping and singular control problems. Furthermore, we describe the form of the optimal strategy by means of viscosity solution techniques and smooth-fit properties on the corresponding system of variational inequalities. Our results are of quasi-explicit nature. From a financial viewpoint, we characterize situations where a firm manager decides optimally to postpone dividend distribution in order to invest in a reversible growth opportunity corresponding to a modern technology. In this paper, a reversible opportunity means that the firm may disinvest from the modern technology and return back to its old technology by receiving some gain compensation. The results of our analysis take qualitatively different forms depending on the parameters values.

Keywords: mixed singular / switching control problem, viscosity solution, smooth-fit property, system of variational inequalities.

4.1 Introduction

Stochastic optimization problems that involve both bounded variation control and/or optimal switching are becoming timely problems in the applied probability literature and more particulary in Mathematical Finance. On one hand, the study of singular stochastic control problems in corporate Finance originates with the research on optimal dividend policy for a firm whose cash reserve follows a diffusion model, see Jeanblanc and Shiryaev [43] and Choulli, Taksar and Zhou [18]. On the other hand, the combined singular / stopping control problems have emerged in target tracking models, see Davis and Zervos [23] and Karatzas, Ocone, Wang and Zervos [46] as well as in Mathematical Finance from firm investment theory. For instance, Guo and Pham [36] have studied the optimal time to activate production and to control it by buying or selling capital while Zervos [68] has applied this type of mixed problems in the field of real options theory. Finally, the theory of investment under uncertainty for a firm that can operate a production activity in different modes has led to optimal switching problems which have received a lot of attention in recent years from the applied mathematics community, see Brekke and Oksendal [12], Duckworth and Zervos [27], Hamadène and Jeanblanc [37], Ly Vath and Pham [51].

In this paper, we consider a combined stochastic control problem that has emerged in a recent paper by Décamps and Villeneuve [24] with the study of the interactions between dividend policy and investment under uncertainty. These authors have studied the interaction between dividend policy and irreversible investment decision in a growth opportunity. Our aim is to extend this work by relaxing the irreversible feature of the growth opportunity. In other words, we shall consider a firm with a technology in place that has the opportunity to invest in a new technology that increases its profitability. The firm self-finances the opportunity cost on its cash reserve. Once installed, the manager can decide to return back to the old technology by receiving some cash compensation. The mathematical formulation of this problem leads to a combined singular control/switching control for a one dimensional diffusion process. The diffusion process may take two regimes, old or new, that are switched at stopping times decisions. Within a regime, the manager has to choose a dividend policy that maximizes the expected value of all payouts until bankruptcy or regime transition. The transition from one regime to another incurs a cost or a benefit. The problem is to find the optimal mixed strategy that maximizes the expected returns.

Our analysis is rich enough to address several important questions that have arisen recently in the real option literature [1]. What is the effect of financing constraints on investment decision? When is it optimal to postpone dividends distribution in order to invest? Basically, two assumptions in the real option theory are that the investment decision is made independently of the financial structure of the investment firm and also that the cash process generated by the investment is independent of any managerial decision. In contrast, our model studies the investment under uncertainty with the following set of assumptions.

[1] See the book of Dixit and Pyndick [26] for an overview of this literature.

The firm is cash constrained and must finance its investments on its cash benefits, and the cash process generated by the investment depends only on the managerial decision to pay or not dividends, to quit or not the project. Our major finding is to characterize the natural intuition that the manager will delay dividend payments if the investment is sufficiently valuable.

As usual in stochastic control theory, the problem developed in this paper leads via the dynamic programming principle to a Hamilton Jacobi Bellman equation which forms in this paper a system of coupled variational inequalities. Therefore, a classical approach based on a verification theorem fails since it is very difficult to guess the shape of both the value function and optimal strategy. To circumvent this difficulty, we use a viscosity solution approach and uniqueness result combined with smooth-fit properties for determining the solution to the HJB system. As by product, we also determine the shape of switching regions. Our findings take qualitatively different forms depending on both the profit rates of each technology and transition costs.

The paper is organized as follows. We formulate the combined stochastic control problem in Section 4.2. In Section 4.3, we characterize by means of viscosity solutions, the system of variational inequalities satisfied by the value function, and we also state some regularity properties. Section 4.4 is devoted to qualitative results concerning the switching regions and in Section 4.5 we give the quasi-explicit computation and description of the value function and the optimal strategies.

4.2 Model formulation : a mixed switching/singular control problem

We consider a firm whose activities generate cash process. The manager of the firm acts in the best interest of its shareholders and maximizes the expected present value of dividends up to bankruptcy when the cash reserve becomes negative. The firm has at any time the possibility to invest in a modern technology that increases the drift of the cash from μ_0 to μ_1 without affecting the volatility σ. This growth opportunity requires a fixed cost $g > 0$ self-financed by the cash reserve. Moreover, we consider a reversible investment opportunity for the firm : the manager can decide to return back to the old technology by receiving some fixed gain compensation $(1-\lambda)g$, with $0 < \lambda < 1$.

The mathematical formulation of this mixed singular/switching control problem is as follows. Let W be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.
- A strategy decision for the firm is a singular/switching control $\alpha = (Z, (\tau_n)_{n\geq 1}) \in \mathcal{A}$ where $Z \in \mathcal{Z}$, the set of $\mathbb{F}$-adapted cadlag nondecreasing processes, $Z_{0^-} = 0$, $(\tau_n)_n$ is an increasing sequence of stopping times, $\tau_n \to \infty$. Z represents the total amount of dividends paid until time t, (τ_n) the switching technology (regimes) time decisions. By convention regime $i = 0$ represents the old technology and $i = 1$ the modern technology.

- Starting from an initial state $(x,i) \in \mathbb{R} \times \{0,1\}$ for the cash-regime value, and given a control $\alpha \in \mathcal{A}$, the dynamics of the cash reserve process of a firm is governed by :

$$dX_t = \mu_{I_t} dt + \sigma dW_t - dZ_t - dK_t, \quad X_{0^-} = x, \tag{4.2.1}$$

where :

$$\begin{aligned} I_t &= \sum_{n\geq 0} \left(i 1_{\tau_{2n}\leq t<\tau_{2n+1}} + (1-i) 1_{\tau_{2n+1}\leq t<\tau_{2n+2}} \right), \quad I_{0^-} = i \qquad (4.2.2) \\ K_t &= \sum_{n\geq 0} \left(g_{i,1-i} 1_{\tau_{2n+1}\leq t<\tau_{2n+2}} + g_{1-i,i} 1_{\tau_{2n+2}\leq t<\tau_{2n+3}} \right), \end{aligned}$$

with

$$\begin{aligned} &0 \leq \mu_0 < \mu_1 \qquad \sigma > 0, \\ &g_{01} = g > 0, \qquad g_{10} = -(1-\lambda) g < 0, \quad 0 < \lambda < 1. \end{aligned}$$

(Here we used the convention $\tau_0 = 0$). We denote by $(X^{x,i}, I^i)$ the solution to (4.2.1)-(4.2.2) (as usual, we omit the dependance in the control α when there is no ambiguity). The time of strict bankruptcy is defined as

$$T = T^{x,i,\alpha} = \inf \left\{ t \geq 0 \ : X_t^{x,i} < 0 \right\},$$

and we set by convention $X_t^{x,i} = X_T^{x,i}$ for $t \geq T$. Thus, for $t \in [T \wedge \tau_{2n}, T \wedge \tau_{2n+1})$, the cash reserve $X^{x,i}$ is in technology i (its drift term is μ_i), while for $t \in [T \wedge \tau_{2n+1}, T \wedge \tau_{2n+2})$, $X^{x,i}$ is in technology $1-i$ (its drift term is μ_{1-i}). Moreover,

$$\begin{aligned} X^{x,i}_{T\wedge\tau_{2n+1}} &= X^{x,i}_{(T\wedge\tau_{2n+1})^-} - g_{i,1-i} \quad \text{on} \quad \{\tau_{2n+1} < T\} \\ X^{x,i}_{T\wedge\tau_{2n+2}} &= X^{x,i}_{(T\wedge\tau_{2n+2})^-} - g_{1-i,i} \quad \text{on} \quad \{\tau_{2n+2} < T\}. \end{aligned}$$

The optimal firm value is

$$v_i(x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t \right], \quad x \in \mathbb{R}, \ i = 0,1. \tag{4.2.3}$$

Notice that v_i is nonnegative, and $v_i(x) = 0$ for $x < 0$. Since $T = T^{x,i,\alpha}$ is obviously nondecreasing in x, the value functions v_i are clearly nondecreasing.

Remark 4.2.1 For any $x > 0$, $i = 0,1$, and given an arbitrary control $\alpha = (Z, (\tau_n)_{n\geq 1}) \in \mathcal{A}$, let us consider the control $\tilde{\alpha} = (\tilde{Z}, (\tau_n)_{n\geq 1}) \in \mathcal{A}$ with $\tilde{Z}_t = Z_t + \eta$, for $t \geq 0$, and $0 < \eta < x$. Then, by stressing the dependence of the state process on the control, we have $X_t^{x,i,\tilde{\alpha}} = X_t^{x-\eta,i,\alpha}$ for $0 \leq t < T^{x,i,\tilde{\alpha}} = T^{x-\eta,i,\alpha}$. We deduce

$$v_i(x) \geq \mathbb{E}\left[\int_0^{(T^{x,i,\tilde{\alpha}})^-} e^{-\rho t} d\tilde{Z}_t \right] = \mathbb{E}\left[\int_0^{(T^{x-\eta,i,\alpha})^-} e^{-\rho t} dZ_t \right],$$

which implies from the arbitrariness of α :

$$v_i(x) \geq v_i(x-\eta), \quad 0 < \eta < x.$$

This shows in particular that v is increasing on $(0,\infty)$.

4.3 Dynamic programming and general properties on the value functions

We first introduce some notations. We denote by $R^{x,i}$ the cash reserve in absence of dividends distribution and in regime i, i.e. the solution to

$$dR_t \;=\; \mu_i dt + \sigma dW_t, \quad R_0 \;=\; x. \tag{4.3.1}$$

The associated second order differential operator is denoted $\mathcal{L}_i$:

$$\mathcal{L}_i\varphi(x) \;=\; \mu_i\varphi'(x) + \frac{1}{2}\sigma^2\varphi''(x).$$

In view of the dynamic programming principle, recalled below (see (4.3.20)), we formally expect that the value functions v_i, $i = 0, 1$, satisfy the system of variational inequalities :

$$\min\left[\rho v_i(x) - \mathcal{L}_i v_i(x), v_i'(x) - 1, v_i(x) - v_{1-i}(x - g_{i,1-i})\right] = 0, \quad x > 0, \; i = 0, 1. \tag{4.3.2}$$

This statement will be proved rigorously later by means of viscosity solutions. For the moment, we state a standard first result for this system of PDE.

Proposition 4.3.1 *Suppose that φ_i, $i = 0, 1$, are two smooth functions on $(0, \infty)$ s.t. $\varphi_i(0^+) := \lim_{x\downarrow 0}\varphi_i(x) \geq 0$, and*

$$\min\left[\rho\varphi_i(x) - \mathcal{L}_i\varphi_i(x), \varphi_i'(x) - 1, \varphi_i(x) - \varphi_{1-i}(x - g_{i,1-i})\right] \geq 0, \; x > 0, \; i = 0, 1, \tag{4.3.3}$$

where we set by convention $\varphi_i(x) = 0$ for $x < 0$. Then, we have $v_i \leq \varphi_i$, $i = 0, 1$.

Proof. Given an initial state-regime value $(x, i) \in \mathbb{R}_+ \times \{0, 1\}$, take an arbitrary control α $= (Z, (\tau_n), n \geq 1) \in \mathcal{A}$, and set for $m > 0$, $\theta_{m,n} = \inf\{t \geq T \wedge \tau_{2n} \; : X_t^{x,i} \geq m\} \nearrow \infty$ a.s. when m goes to infinity. Apply then Itô's formula to $e^{-\rho t}\varphi_i(X_t^{x,i})$ between the stopping times $T \wedge \tau_{2n}$ and $\tau_{m,2n+1} : = T \wedge \tau_{2n+1} \wedge \theta_{m,n}$. Notice that for $T \wedge \tau_{2n} \leq t < \tau_{m,2n+1}$, $X_t^{x,i}$ stays in regime i. Then, we have

$$\begin{aligned} e^{-\rho\tau_{m,2n+1}}\varphi_i(X^{x,i}_{\tau^-_{m,2n}}) \;&=\; e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}}) + \int_{T\wedge\tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}(-\rho\varphi_i + \mathcal{L}_i\varphi_i)(X_t^{x,i})dt \\ &+ \int_{T\wedge\tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}\sigma\varphi_i'(X_t^{x,i})dW_t - \int_{T\wedge\tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}\varphi_i'(X_t^{x,i})dZ_t^c \\ &+ \sum_{T\wedge\tau_{2n}\leq t<\tau_{m,2n+1}} e^{-\rho t}\left[\varphi_i(X_t^{x,i}) - \varphi_i(X_{t^-}^{x,i})\right], \end{aligned} \tag{4.3.4}$$

where Z^c is the continuous part of Z. We make the convention that when $T \leq \tau_n$, $(T \wedge \theta)^-$ $= T$ for all stopping time $\theta > \tau_n$ a.s., so that (4.3.4) holds true a.s. for all n, m (recall that $\varphi_i(X_T^{x,i}) = 0$). Since $\varphi_i' \geq 1$, we have by the mean-value theorem $\varphi_i(X_t^{x,i}) - \varphi_i(X_{t^-}^{x,i})$ $\leq X_t^{x,i} - X_{t^-}^{x,i} = -(Z_t - Z_{t^-})$ for $T \wedge \tau_{2n} \leq t < \tau_{m,2n+1}$. By using also the supersolution inequality of φ_i, taking expectation in the above Itô's formula, and noting that the integrand in the stochastic integral term is bounded by a constant (depending on m), we have

$$\begin{aligned}\mathbb{E}\left[e^{-\rho\tau_{m,2n+1}}\varphi_i(X^{x,i}_{\tau^-_{m,2n+1}})\right] &\leq \mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right] - \mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}dZ^c_t\right] \\ &\quad - \mathbb{E}\left[\sum_{T\wedge\tau_{2n}\leq t<\tau_{m,2n+1}} e^{-\rho t}(Z_t - Z_{t^-})\right],\end{aligned}$$

and so

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{\tau^-_{m,2n+1}} e^{-\rho t}dZ_t + e^{-\rho\tau_{m,2n+1}}\varphi_i(X^{x,i}_{\tau^-_{m,2n+1}})\right]$$

By sending m to infinity, with Fatou's lemma, we obtain :

$$\begin{aligned}&\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right] \\ \geq\ &\mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{(T\wedge\tau_{2n+1})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n+1})}\varphi_i(X^{x,i}_{(T\wedge\tau_{2n+1})^-})\right]. \qquad (4.3.5)\end{aligned}$$

Now, as $\varphi_i(x) \geq \varphi_{1-i}(x - g_{i,1-i})$ and recalling $X^{x,i}_{T\wedge\tau_{2n+1}} = X^{x,i}_{(T\wedge\tau_{2n+1})^-} - g_{i,1-i}$ on $\{\tau_{2n+1} < T\}$, we have

$$\begin{aligned}\varphi_i(X^{x,i}_{(T\wedge\tau_{2n+1})^-}) &\geq \varphi_{1-i}(X^{x,i}_{(T\wedge\tau_{2n+1})^-} - g_{i,1-i}) \\ &= \varphi_{1-i}(X^{x,i}_{(T\wedge\tau_{2n+1})}) \quad \text{on } \{\tau_{2n+1} < T\}. \qquad (4.3.6)\end{aligned}$$

Moreover, notice that φ_i is nonnegative as $\varphi_i(0^+) \geq 0$ and $\varphi_i' \geq 1$. Hence, since $\varphi_{1-i}(X^{x,i}_{(T\wedge\tau_{2n+1})})$ $= \varphi_{i-1}(X^{x,i}_T) = 0$ on $\{T \leq \tau_{2n+1}\}$, we see that inequality (4.3.6) also holds on $\{T \leq \tau_{2n+1}\}$ and so a.s. Therefore, plugging into (4.3.5), we have

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{(T\wedge\tau_{2n+1})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n+1})}\varphi_{1-i}(X^{x,i}_{T\wedge\tau_{2n+1}})\right].$$

Similarly, we have from the supersolution inequality of φ_{1-i} :

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n+1})}\varphi_{1-i}(X^{x,i}_{T\wedge\tau_{2n+1}})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_{2n+1}}^{(T\wedge\tau_{2n+2})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n+2})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n+2}})\right].$$

By iterating these two previous inequalities for all n, we then obtain

$$\begin{aligned}\varphi_i(x) &\geq \mathbb{E}\left[\int_0^{(T\wedge\tau_{2n})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right], \\ &\geq \mathbb{E}\left[\int_0^{(T\wedge\tau_{2n})^-} e^{-\rho t}dZ_t\right], \quad \forall n \geq 0,\end{aligned}$$

since φ_i is nonnegative. By sending n to infinity, we obtain the required result from the arbitrariness of the control α. □

As a corollary, we show a linear growth condition on the value functions.

Corollary 4.3.1 *We have,*

$$v_0(x) \ \leq \ x + \frac{\mu_1}{\rho}, \qquad v_1(x) \ \leq \ x + \frac{\mu_1}{\rho} + (1-\lambda)g, \quad x > 0. \tag{4.3.7}$$

Proof. We set $\varphi_0(x) = x + \frac{\mu_1}{\rho}$, $\varphi_1(x) = x + \frac{\mu_1}{\rho} + (1-\lambda)g$, on $(0,\infty)$, and $\varphi_i(x) = 0$ for $x < 0$. A straightforward computation shows that we have the supersolution properties for φ_i, $i = 0, 1$:

$$\begin{aligned} \min\left[\rho\varphi_0(x) - \mathcal{L}_0\varphi_0(x), \varphi_0'(x) - 1, \varphi_0(x) - \varphi_1(x-g)\right] \ &\geq \ 0, \quad x > 0, \\ \min\left[\rho\varphi_1(x) - \mathcal{L}_1\varphi_1(x), \varphi_1'(x) - 1, \varphi_1(x) - \varphi_0(x + (1-\lambda)g)\right] \ &\geq \ 0, \quad x > 0. \end{aligned}$$

We then conclude from Proposition 4.3.1. □

The next result states the initial-boundary data for the value functions.

Proposition 4.3.2 *1) The value function v_0 is continuous on $(0,\infty)$ and satisfies*

$$v_0(0^+) \ := \ \lim_{x \downarrow 0} v_0(x) \ = \ 0. \tag{4.3.8}$$

2) The value function v_1 satisfies

$$v_1(0^+) \ := \ \lim_{x \downarrow 0} v_1(x) \ = \ v_0((1-\lambda)g). \tag{4.3.9}$$

Proof. 1) a) We first prove (4.3.8). For $x > 0$, let us consider the drifted Brownian $R^{x,1}$, defined in (4.3.1), and denote $\theta_0 = \inf\{t \geq 0 \ : R_t^{x,1} = 0\}$. It is well-known that :

$$\mathbb{E}\left[\sup_{0 \leq t \leq \theta_0} R_t^{x,1}\right] \ \to \ 0, \quad \text{as } x \downarrow 0. \tag{4.3.10}$$

We also have

$$\sup_{0 \leq t \leq \theta_0} R_t^{x,1} \ \downarrow \ 0, \quad a.s. \quad \text{as } x \downarrow 0. \tag{4.3.11}$$

Fix some $r > 0$, and denote $\theta_r = \inf\{t \geq 0 \ : R_t^{x,1} = r\}$. It is also well-known that

$$\mathbb{P}\left[\theta_0 > \theta_r\right] \ \to \ 0, \quad \text{as } x \downarrow 0. \tag{4.3.12}$$

Let $\alpha = (Z, (\tau_n)_{n \geq 1})$ be an arbitrary policy in $\mathcal{A}$, and denote $\eta = T \wedge \theta_r = T^{x,0,\alpha} \wedge \theta_r$. Since $\mu_0 < \mu_1$ and $g_{01} + g_{10} > 0$, we notice that $X_t^{x,0} \leq R_t^{x,1} - Z_t \leq R_t^{x,1}$ for all $t \geq 0$. Hence $T \leq \theta_0$, $Z_t \leq R_t^{x,1}$ for $t < T$, and in particular $Z_{\eta^-} \leq R_\eta^{x,1}$. We then write :

$$\begin{aligned} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] \ &= \ \mathbb{E}\left[\int_0^{\eta^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[1_{T > \eta} \int_\eta^{T^-} e^{-\rho t} dZ_t\right] \\ &\leq \ \mathbb{E}\left[Z_{\eta^-}\right] + \mathbb{E}\left[1_{T > \eta} \mathbb{E}\left[\int_\eta^{T^-} e^{-\rho t} dZ_t \middle| \mathcal{F}_\eta\right]\right] \\ &\leq \ \mathbb{E}\left[R_\eta^{x,1}\right] + \mathbb{E}\left[1_{T > \theta_r} e^{-\gamma\eta} v_0(X_{\eta^-}^{x,0})\right], \end{aligned} \tag{4.3.13}$$

where we also used in the last inequality the definition of the value function v_0. Now, since $T \leq \theta_0$, we have $\eta \leq \theta_0$. Moreover, since v_0 is nondecreasing and $\eta \leq \theta_r$, we have $v_0(X^{x,0}_{\eta^-})$ $\leq v_0(r)$. Thus, inequality (4.3.13) yields

$$0 \;\leq\; v_0(x) \;\;\leq\;\; \mathbb{E}\left[\sup_{0\leq t\leq\theta_0} R^{x,1}_t\right] + v_0(r)\mathbb{P}[\theta_0 > \theta_r] \;\;\longrightarrow\;\; 0, \quad \text{as } x \downarrow 0, \tag{4.3.14}$$

from (4.3.10)-(4.3.12). This proves $v_0(0^+) = 0$.

b) We next prove the continuity of v_0 at any $y > 0$. Let $\alpha = (Z, (\tau_n)_{n\geq1}) \in \mathcal{A}$, $X^{y,0}$ be the corresponding process and $T = T^{y,0,\alpha}$ its bankruptcy time. According to (4.3.10) and (4.3.12), given a fixed $r > 0$, for any arbitrary small $\varepsilon > 0$, one can find $0 < \delta < y$ s.t. for $0 < x < \delta$,

$$\mathbb{E}\left[\sup_{0\leq t\leq\theta_0} R^{x,1}_t\right] + v_0(r)\mathbb{P}[\theta_0 > \theta_r] \;\;\leq\;\; \varepsilon,$$

Then, following the same lines of proof as for (4.3.13)-(4.3.14), we show

$$\mathbb{E}\left[\int_\theta^{T^-} e^{-\rho t} dZ_t\right] \;\;\leq\;\; \varepsilon, \tag{4.3.15}$$

for any $0 < x < \delta$ and stopping time θ s.t. $X^{y,0}_\theta \leq x$. Given $0 < x < \delta$, consider the state process $X^{y-x,0}$ starting from $y - x$ in regime 0, and controlled by α. Denote θ its bankruptcy time, i.e. $\theta = T^{y-x,0,\alpha} = \inf\{t \geq 0 \;\;: X^{y-x,0}_t < 0\}$. Notice that $X^{y-x,0}_t = X^{y,0}_t - x$ for $t \leq \theta \leq T$, and so

$$X^{y,0}_\theta \;\;=\;\; X^{y-x,0}_\theta + x \;\leq\; x.$$

From (4.3.15), we then have

$$\begin{aligned}
\mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] \;\;&=\;\; \mathbb{E}\left[\int_0^{\theta^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[\int_\theta^{T^-} e^{-\rho t} dZ_t\right] \\
&\leq\;\; v_0(y - x) + \varepsilon.
\end{aligned}$$

From the arbitrariness of α, and recalling that v_0 is nondecreasing, this implies

$$0 \;\leq\; v_0(y) - v_0(y - x) \;\;\leq\;\; \varepsilon,$$

which shows the continuity of v_0.

2) Given an arbitrary control $\alpha = (Z, (\tau_n)_{n\geq1}) \in \mathcal{A}$, let us consider the control $\tilde{\alpha} = (\tilde{Z}, (\tilde{\tau}_n)_{n\geq1}) \in \mathcal{A}$ defined by $\tilde{Z} = Z$, $\tilde{\tau}_1 = 0$, $\tilde{\tau}_n = \tau_{n-1}$, $n \geq 2$. Then, for all $x > 0$, and by stressing the dependence of the state process on the control, we have $X^{x,1,\tilde{\alpha}}_t = X^{x+(1-\lambda)g,0,\alpha}_t$ for $0 \leq t < T^{x,1,\tilde{\alpha}} = T^{x+(1-\lambda)g,0,\alpha}$. We deduce

$$v_1(x) \;\geq\; \mathbb{E}\left[\int_0^{(T^{x,1,\tilde{\alpha}})^-} e^{-\rho t} d\tilde{Z}_t\right] \;\;=\;\; \mathbb{E}\left[\int_0^{(T^{x+(1-\lambda)g,0,\alpha})^-} e^{-\rho t} dZ_t\right],$$

which implies from the arbitrariness of α :

$$v_1(x) \quad \geq \quad v_0(x + (1-\lambda)g), \quad x > 0. \tag{4.3.16}$$

On the other hand, starting in the regime $i = 1$, for $x \geq 0$, let $\alpha = (Z, (\tau_n)_{n\geq 1})$ be an arbitrary control in $\mathcal{A}$. We denote $T_1 = T \wedge \tau_1 = T^{x,1,\alpha} \wedge \tau_1$, and we write :

$$\mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] \quad = \quad \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[1_{T>\tau_1}\int_{\tau_1}^{T^-} e^{-\rho t} dZ_t\right]. \tag{4.3.17}$$

The first term in the r.h.s. of (4.3.17) is dealt similarly as in (4.3.13)-(4.3.14) : we set $\eta_1 = T_1 \wedge \theta_r$ with $\theta_r = \inf\{t \geq 0 \ : R_t^{x,1} = r\}$ for some fixed $r > 0$, and we notice that $X_t^{x,1} = R_t^{x,1} - Z_t \leq R_t^{x,1}$ for $t < \tau_1$. Hence $T_1 \leq \theta_0 = \inf\{t \geq 0 \ : R_t^{x,1} = 0\}$, and $Z_{\eta_1^-} \leq R_{\eta_1}^{x,1} \leq \sup_{0\leq t\leq \theta_0} R_t^{x,1}$. Then, as in (4.3.13)-(4.3.14), we have :

$$\mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t\right] \quad \leq \quad \mathbb{E}\left[\sup_{0\leq t\leq\theta_0} R_t^{x,1}\right] + v_1(r)\mathbb{P}[\theta_0 > \theta_r]. \tag{4.3.18}$$

For the second term in the r.h.s. of (4.3.17), since there is a change of regime at τ_1 from $i = 1$ to $i = 0$, and by definition of the value function v_0, we have :

$$\begin{aligned}
\mathbb{E}\left[1_{T>\tau_1}\int_{\tau_1}^{T^-} e^{-\rho t} dZ_t\right] \quad &= \quad \mathbb{E}\left[1_{T>\tau_1}\mathbb{E}\left[\int_{\tau_1}^{T^-} e^{-\rho t} dZ_t \middle| \mathcal{F}_{\tau_1}\right]\right] \\
&\leq \quad \mathbb{E}\left[1_{T>\tau_1} e^{-\rho\tau_1} v_0(X_{\tau_1}^{x,1})\right] \\
&\leq \quad \mathbb{E}\left[1_{T>\tau_1} v_0(X_{\tau_1^-}^{x,1} + (1-\lambda)g)\right] \\
&\leq \quad \mathbb{E}\left[v_0\left(\sup_{0\leq t\leq\theta_0} R_t^{x,1} + (1-\lambda)g\right)\right]. \qquad (4.3.19)
\end{aligned}$$

Here, we used in the second inequality the fact that $X_{\tau_1}^{x,1} = X_{\tau_1^-}^{x,1} + (1-\lambda)g$ on $\{\tau_1 < T\}$, and in the last one the observation that $X_t^{x,1} \leq R_t^{x,1}$ for $t < \tau_1$, and $\tau_1 = T_1 \leq \theta_0$ on $\{\tau_1 < T\}$. Hence, by combining (4.3.16)-(4.3.17)-(4.3.18)-(4.3.19), we obtain :

$$\begin{aligned}
& v_0(x + (1-\lambda)g) \\
\leq \quad & v_1(x) \ \leq \ \mathbb{E}\left[\sup_{0\leq t\leq\theta_0} R_t^{x,1}\right] + v_1(r)\mathbb{P}[\theta_0 > \theta_r] + \mathbb{E}\left[v_0\left(\sup_{0\leq t\leq\theta_0} R_t^{x,1} + (1-\lambda)g\right)\right].
\end{aligned}$$

Finally, by using the continuity of v_0, the limits (4.3.10)-(4.3.11)-(4.3.12), as well as the linear growth condition (4.3.7) of v_0, which allows to apply dominated convergence theorem, we conclude that $v_1(0^+) = v_0((1-\lambda)g)$. □

Remark 4.3.1 There is some asymmetry between the two value functions v_0 and v_1. Actually, v_0 is continuous at 0 : $v_0(0^+) = v_0(0^-) = 0$, while it is not the case for v_1, since $v_1(0^+) = v_0((1-\lambda)g) > 0 = v_1(0^-)$: When the reserve process in regime 0 approaches zero, we are ineluctably absorbed by this threshold. On the contrary, in regime 1, when the reserve process approaches zero, we have the possibility to change of regime, which pushes us above the bankruptcy threshold by receiving $(1-\lambda)g$. In particular, at this stage, we do not know yet the continuity of v_1 on $(0,\infty)$. This will be proved in Theorem 4.3.1 as a consequence of the dynamic programming principle. In the sequel, we set by convention $v_i(0) = v_i(0^+)$ for $i = 0,1$.

The following dynamic programming principle holds : for any $(x,i) \in \mathbb{R}_+ \times \{0,1\}$, we have

$$\begin{aligned}(\mathbf{DP}) \;\; v_i(x) \;\; &= \;\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Bigg[\int_0^{(T\wedge\theta\wedge\tau_1)^-} e^{-\rho t} dZ_t \\ &+ \; e^{-\rho(T\wedge\theta\wedge\tau_1)}\Big(v_i(X^{x,i}_{T\wedge\theta}) 1_{T\wedge\theta<\tau_1} + v_{1-i}(X^{x,i}_{\tau_1}) 1_{\tau_1\leq T\wedge\theta}\Big)\Bigg] \qquad (4.3.20)\end{aligned}$$

where θ is any stopping time, possibly depending on $\alpha \in \mathcal{A}$ in (4.3.20).

We then have the PDE characterization of the value functions v_i.

Theorem 4.3.1 *The value functions v_i, $i = 0,1$, are continuous on $(0,\infty)$, and are the unique viscosity solutions with linear growth condition on $(0,\infty)$ and boundary data $v_0(0)$ $= 0$, $v_1(0) = v_0((1-\lambda)g)$ to the system of variational inequalities :*

$$\min\left[\rho v_i(x) - \mathcal{L}_i v_i(x), v_i'(x) - 1, v_i(x) - v_{1-i}(x - g_{i,1-i})\right] = 0, \quad x > 0, \; i = 0,1. \qquad (4.3.21)$$

Actually, we prove some more regularity results on the value functions.

Proposition 4.3.3 *The value functions v_i, $i = 0,1$, are C^1 on $(0,\infty)$. Moreover, if we set for $i = 0,1$:*

$$\begin{aligned}\mathcal{S}_i &= \{x \geq 0 \; : v_i(x) \; = \; v_{1-i}(x - g_{i,1-i})\} \\ \mathcal{D}_i &= \left\{x > 0 \; : v_i'(x) = 1\right\}, \\ \mathcal{C}_i &= (0,\infty) \setminus (\mathcal{S}_i \cup \mathcal{D}_i),\end{aligned}$$

then v_i is C^2 on the open set $\mathcal{C}_i \cup \mathcal{D}_i$ of $(0,\infty)$, and we have in the classical sense

$$\rho v_i(x) - \mathcal{L}_i v_i(x) \;\; = \;\; 0, \quad x \in \mathcal{C}_i.$$

The proofs of Theorem 4.3.1 and Proposition 4.3.3 follow and combine essentially arguments from [36] for singular control, and [57] for switching control, and are postponed to Appendix A and B.

$\mathcal{S}_i$ is the switching region from technology i to $1-i$, $\mathcal{D}_i$ is the dividend region in technology i, and $\mathcal{C}_i$ is the continuation region in technology i. Notice from the boundary conditions on v_1 that $\mathcal{S}_1$ contains 0. We denote $\mathcal{S}_1^* = \mathcal{S}_1 \setminus \{0\}$.

4.4 Qualitative results on the switching regions

4.4.1 Benchmarks

We consider the firm value without investment/disinvestment in technology $i = 0$:

$$\hat{V}_0(x) \quad = \quad \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_0^-} e^{-\rho t} dZ_t\right], \tag{4.4.1}$$

where $T_0 = \inf\{t \geq 0 \ : X_t \leq 0\}$ is the time bankruptcy of the cash reserve in regime 0 :

$$dX_t \quad = \quad \mu_0 dt + \sigma dW_t - dZ_t, \quad X_{0^-} \ = \ x.$$

It is known that $\hat{V}_0$, as the value function of a pure singular control problem, is characterized as the unique continuous viscosity solution on $(0, \infty)$, with linear growth condition to the variational inequality :

$$\min\left[\rho \hat{V}_0 - \mathcal{L}_0 \hat{V}_0 \ , \ \hat{V}_0' - 1\right] \quad = \quad 0, \quad x > 0, \tag{4.4.2}$$

and boundary data

$$\hat{V}_0(0) \quad = \quad 0.$$

Actually, $\hat{V}_0$ is C^2 on $(0, \infty)$ and explicit computations of this standard singular control problem are developed in Shreve, Lehoczky and Gaver [61], Jeanblanc and Shiryaev [43], or Radner and Shepp [59] :

$$\hat{V}_0(x) \quad = \quad \begin{cases} \frac{f_0(x)}{f_0'(\hat{x}_0)} & , 0 \leq x \leq \hat{x}_0 \\ x - \hat{x}_0 + \frac{\mu_0}{\rho}, & x \geq \hat{x}_0, \end{cases}$$

where

$$f_0(x) \quad = \quad e^{m_0^+ x} - e^{m_0^- x}, \quad \hat{x}_0 \ = \ \frac{1}{m_0^+ - m_0^-} \ln\left(\frac{(m_0^+)^2}{(m_0^-)^2}\right),$$

and $m_0^- < 0 < m_0^+$ are roots of the characteristic equation :

$$\rho - \mu_0 m - \frac{1}{2}\sigma^2 m^2 \quad = \quad 0.$$

In other words, this means that the optimal cash reserve process is given by the reflected diffusion process at the threshold $\hat{x}_0$ with an optimal dividend process given by the local time at this boundary. When the firm starts with a cash reserve $x \geq \hat{x}_0$, the optimal dividend policy is to distribute immediately the amount $x - \hat{x}_0$ and then follows the dividend policy characterized by the local time.

As a second benchmark, we consider the firm value problem in technology $i = 1$ with nonnegative constant liquidation value L to be fixed later :

$$w_1^L(x) \quad = \quad \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} L\right], \tag{4.4.3}$$

$T_1 = \inf\{t \geq 0 \ : X_t \leq 0\}$ is the time bankruptcy of the cash reserve in regime 1 :

$$dX_t \;=\; \mu_1 dt + \sigma dW_t - dZ_t, \quad X_{0^-} \;=\; x.$$

Again, as value function of a pure singular control problem, w_1^L is characterized as the unique continuous viscosity solution on $(0,\infty)$, with linear growth condition to the variational inequality :

$$\min\left[\rho w_1^L - \mathcal{L}_1 w_1^L \ , \ (w_1^L)' - 1\right] \;=\; 0, \quad x > 0, \tag{4.4.4}$$

and boundary data

$$w_1^L(0) \;=\; L. \tag{4.4.5}$$

Actually, w_1^L is C^2 on $(0,\infty)$ and explicit computations of this singular control problem are developed in Boguslavskaya [9] :

• If $L \geq \frac{\mu_1}{\rho}$, then :

$$w_1^L(x) \;=\; x + L, \quad x \geq 0.$$

The optimal strategy is to distribute the initial cash reserve immediately, and so to liquidate the firm at $X_t = 0$ by changing of technology to regime $i = 0$ and receiving L.

• If $L < \frac{\mu_1}{\rho}$, then

$$w_1^L(x) \;=\; \begin{cases} \frac{1 - Lh_1'(\hat{x}_1)}{f_1'(\hat{x}_1)} f_1(x) + Lh_1(x) & , 0 \leq x \leq x_1^L \\ x - x_1^L + \frac{\mu_1}{\rho}, & x \geq x_1^L, \end{cases}$$

with

$$f_1(x) \;=\; e^{m_1^+ x} - e^{m_1^- x}, \quad h_1(x) \;=\; e^{m_1^- x},$$

$m_1^- < 0 < m_1^+$, the roots of the characteristic equation :

$$\rho - \mu_1 m - \frac{1}{2}\sigma^2 m^2 \;=\; 0,$$

and x_1^L the solution to

$$L\frac{h_1(x)f_1'(x) - h_1'(x)f_1(x)}{f_1'(x)} + \frac{f_1(x)}{f_1'(x)} \;=\; \frac{\mu_1}{\rho}.$$

The optimal cash reserve process is given by the reflected diffusion process at the threshold x_1^L with an optimal dividend process given by the local time at this boundary. When the firm starts with a cash reserve $x \geq x_1^L$, the optimal dividend policy is to distribute immediately the amount $x - x_1^L$ and then follows the dividend policy characterized by the local time. In the sequel, we shall denote

$$\hat{V}_1 \;=\; w_1^L \quad \text{and} \quad \hat{x}_1 \;=\; x_1^L \qquad \text{when} \quad L = \hat{V}_0((1-\lambda)g).$$

$L = \hat{V}_0((1-\lambda)g)$ is the minimal received liquidation value when one switches to regime 0 at $x = 0$ and do not switch anymore.

Remark 4.4.1 It is known (see e.g.[9]) that $\hat{V}_0$ and w_1^L are concave on $(0, \infty)$. As a consequence, $\hat{V}_0$ and w_1^L are globally Lipschitz since their first derivatives are bounded near zero. By convention, we set $\hat{V}_0(x) = w_1^L(x) = 0$ for $x < 0$.

Remark 4.4.2 We have $v_0 \geq \hat{V}_0$ and $v_1 \geq \hat{V}_1$ on $(0, \infty)$. This is rather clear since the class of controls over which maximization is taken in $\hat{V}_0$ and $\hat{V}_1$ is included in the class of controls of v_0 and v_1. This may be justified more rigorously by a maximum principle argument and by noting that v_0 and v_1 are (viscosity) supersolution to the variational inequality satisfied respectively by $\hat{v}_0$ and $\hat{V}_1$, with the same boundary data.

We first show the intuitive result that the value function for the dividend policy problem is nondecreasing in the rate of return of the cash reserve.

Lemma 4.4.1

$$\hat{V}_1(x) \;\; \geq \;\; \hat{V}_0(x + (1-\lambda)g), \quad \forall x \geq 0.$$

Proof. We set $w_1(x) = \hat{V}_1(x - (1-\lambda)g)$ for $x \geq (1-\lambda)g$. From (4.4.4), we see that $\hat{w}_1$ satisfies on $[(1-\lambda)g, \infty)$:

$$\begin{aligned} w_1'(x) &= \hat{V}_1'(x - (1-\lambda)g) \; \geq \; 1 \\ (\rho w_1 - \mathcal{L}_0 w_1)(x) &= (\rho - \mathcal{L}_1 \hat{V}_1 + (\mu_1 - \mu_0)\hat{V}_1')(x - (1-\lambda)g) \; > \; 0, \end{aligned}$$

since $\mu_1 > \mu_0$ and $\hat{V}_1$ is increasing. Moreover, $w_1((1-\lambda)g) = \hat{V}_1(0) = \hat{V}_0((1-\lambda)g)$. By standard maximum principle on the variational inequality (4.4.2), we deduce that $w_1 \geq \hat{V}_0$ on $[(1-\lambda)g, \infty)$, which implies the required result. □

The next result specifies conditions under which the value function in the old technology is larger than the value function in the modern technology after paying the switching cost from the old to the modern regimes.

Lemma 4.4.2 *Suppose that* $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$. *Then,*

$$\hat{V}_0(x) \; \geq \; \hat{V}_1(x-g), \quad \forall x \geq 0, \quad \textit{if and only if} \quad \frac{\mu_1 - \mu_0}{\rho} \; \leq \; \hat{x}_1 + g - \hat{x}_0.$$

Proof. Similar arguments as in Lemma 2.1 in Decamps and Villeneuve [24]. □

Remark 4.4.3 Using the same argument as in the proof of Lemma 4.4.1, the above Lemma shows also that if $\frac{\mu_1 - \mu_0}{\rho} > \hat{x}_1 + g - \hat{x}_0$, then there exists $\hat{x}_{01} \geq g$ s.t.

$$\max\Big(\hat{V}_0(x), \hat{V}_1(x-g)\Big) \;\; = \;\; \begin{cases} \hat{V}_0(x), & x \leq \hat{x}_{01} \\ \hat{V}_1(x-g), & x > \hat{x}_{01} \end{cases}$$

4.4.2 Preliminary results on the switching regions

In this section, we shall state some preliminary qualitative results concerning the switching regions.

Lemma 4.4.3 *If $x \in \mathcal{S}_i$ then $x - g_{i,1-i} \notin \mathcal{S}_{1-i}$.*

Proof. Since $v_i(x) > v_i(x - \lambda g)$ for every $x > 0$ and $i \in \{0,1\}$, we have for $x \in \mathcal{S}_i$,

$$v_{1-i}(x - g_{i,1-i}) \;=\; v_i(x) \;>\; v_i(x - \lambda g) \;=\; v_i(x - g_{i,1-i} - g_{1-i,i}).$$

Therefore, $x - g_{i,1-i} \notin \mathcal{S}_{1-i}$ for $x \in \mathcal{S}_i$. □

Let us recall the notation $\mathcal{S}_i^* = \mathcal{S}_i \setminus \{0\}$. We have the following inclusion :

Lemma 4.4.4 $\mathcal{S}_1^* \subset \mathcal{D}_1$.

Proof. We make a proof by contradiction by assuming that there exists some $x \in \mathcal{S}_1^* \setminus \mathcal{D}_1$. According to Proposition 4.3.3, we have $v_0'(x + (1-\lambda)g) = v_1'(x) > 1$, and so $x + (1-\lambda)g \notin \mathcal{D}_0$. Applying Lemma 4.4.3 with $i = 1$ implies $x + (1-\lambda)g \in \mathcal{C}_0$. Therefore,

$$\begin{aligned}
\rho v_1(x) - \mathcal{L}_1 v_1(x) &= \rho v_1(x) - \mathcal{L}_0 v_1(x) + (\mu_0 - \mu_1) v_1'(x) \\
&= \rho v_0(x + (1-\lambda)g) - \mathcal{L}_0 v_0(x + (1-\lambda)g) + (\mu_0 - \mu_1) v_1'(x) \\
&= (\mu_0 - \mu_1) v_1'(x) \qquad \text{since} \quad x + (1-\lambda)g \in \mathcal{C}_0 \\
&< 0,
\end{aligned}$$

which contradicts Theorem 4.3.1. □

We now introduce the following definition.

Definition 4.4.1 *y is a left boundary of the closed set $\mathcal{D}_i$ if there is some $\delta > 0$ such that $y - \varepsilon$ does not belong to $\mathcal{D}_i$ for every $0 < \varepsilon < \delta$.*

Lemma 4.4.5 *Let $y > 0$ be a left boundary of $\mathcal{D}_i$.*

- *If there is some $\varepsilon > 0$ such that $(y - \varepsilon, y) \subset \mathcal{C}_i$, then $v_i(y) = \dfrac{\mu_i}{\rho}$.*

- *If not, $v_i(y) = \dfrac{\mu_{1-i}}{\rho}$.*

Proof. Since y is a left boundary of $\mathcal{D}_i$, there is some $\varepsilon > 0$ such that $(y - \varepsilon, y) \subset \mathcal{C}_i \cup \mathcal{S}_i$. Therefore, two cases have to be considered.
⋆ *Case 1*: If $(y - \varepsilon, y) \subset \mathcal{C}_i$. Then, according to Proposition 4.3.3, v_i is twice differentiable at x, for $y - \varepsilon < x < y$ and satisfies $v_i'(y) = 1$ and $v_i''(y) = 0$. Therefore, we have

$$0 \;=\; \rho v_i(x) - \mathcal{L}_i v_i(x) \;=\; \rho v_i(x) - \mu_i v_i'(x) - \frac{\sigma^2}{2} v_i''(x).$$

By sending x to y, we obtain that $v_i(y) = \frac{\mu_i}{\rho}$.

$\star$ *Case 2* : If not, there is an increasing sequence $(y_n)_n$ valued in $\mathcal{S}_i$, and converging to y which therefore belongs to $\mathcal{S}_i$. We then have $v_i(y_n) = v_{1-i}(y_n - g_{i,1-i})$ and also $v_i^{'}(y_n) > 1$ for n great enough since y is a left boundary of $\mathcal{D}_i$. Thus, $y_n - g_{i,1-i} \notin \mathcal{D}_{1-i}$. Moreover, according to Lemma 4.4.3, we also have $y_n - g_{i,1-i} \notin \mathcal{S}_{1-i}$ and therefore, $y_n - g_{i,1-i} \in \mathcal{C}_{1-i}$ or equivalently

$$\rho v_{1-i}(y_n - g_{i,1-i}) - \mathcal{L}_{1-i} v_{1-i}(y_n - g_{i,1-i}) = 0.$$

By letting n tends to ∞, we obtain $v_{1-i}(y - g_{i,1-i}) = \frac{\mu_{1-i}}{\rho}$. Since $y \in \mathcal{S}_i$, this implies $v_i(y) = v_{1-i}(y - g_{i,1-i}) = \frac{\mu_{1-i}}{\rho}$.

$\square$

The next result shows that the switching region from modern technology $i = 1$ to the old technology $i = 0$ is either reduced to the zero threshold or to the entire state reserve domain $\mathbb{R}_+$, depending on the gain $(1 - \lambda)g$ for switching from regime 1 to regime 0.

Proposition 4.4.1 *The two following cases arise :*

(i) If $v_0((1 - \lambda)g) < \frac{\mu_1}{\rho}$ *then* $\mathcal{S}_1 = \{0\}$.

(ii) If $v_0((1 - \lambda)g) \geq \frac{\mu_1}{\rho}$ *then* $\mathcal{S}_1 = \mathcal{D}_1 = \mathbb{R}_+$.

Proof. (i) Assume $v_0((1 - \lambda)g) < \frac{\mu_1}{\rho}$.
We shall make a proof by contradiction by considering the existence of some $x_0 \in S_1^*$. By Lemma 4.4.4, one can introduce the finite nonnegative number

$$\underline{\mathrm{x}} = \inf\{y > 0 \ : [y, x_0] \subset \mathcal{D}_1\}.$$

Hence, $\underline{\mathrm{x}}$ is a left boundary of $\mathcal{D}_1$. Moreover, Lemma 4.4.5 gives $v_1(\underline{\mathrm{x}}) = \frac{\mu_1}{\rho}$ or $\frac{\mu_0}{\rho}$.

1. We first check that $\underline{\mathrm{x}} > 0$. If not, we would have: $v_1(y) = y + v_0((1 - \lambda)g)$ for any $0 < y < x_0$. But, in this case, we have for $0 < y < x_0$,

$$\rho v_1(y) - \mathcal{L}_1 v_1(y) = \rho(y + v_0((1 - \lambda)g)) - \mu_1.$$

   Therefore, under the assumption (i), $\rho v_1(y) - \mathcal{L}_1 v_1(y) < 0$ for y small enough which is a contradiction.

2. We now prove that $v_1(\underline{\mathrm{x}}) = \frac{\mu_1}{\rho}$. To see this, we shall show that the closed set $\mathcal{D}_1$ is an interval of $\mathbb{R}_+$. Let $a, b \in \mathcal{D}_1$ with $a < b$, we want to show that $(a, b) \subset \mathcal{D}_1$. If

not, from Lemma 4.4.4, we can find a subinterval (c, d) with $c, d \in \mathcal{D}_1$ and $(c, d) \subset \mathcal{C}_1$. But, for $c < x < d$, we have

$$0 \quad = \quad \rho v_1(x) - \mathcal{L}_1 v_1(x) \;=\; \rho v_1(x) - \mu_1 v_1'(x) - \frac{\sigma^2}{2} v_1''(x).$$

By sending x to c and d, we obtain that $v_1(c) = v_1(d) = \dfrac{\mu_1}{\rho}$ which contradicts the fact that v_1 is strictly increasing. Since $\mathcal{D}_1$ is an interval of $\mathbb{R}_+$, we have $\underline{\mathrm{x}} = \inf \mathcal{D}_1$. Thus, recalling that $\underline{\mathrm{x}} > 0$, we can find from Lemma 4.4.4, some $\varepsilon > 0$ such that $(\underline{\mathrm{x}} - \varepsilon, \underline{\mathrm{x}}) \subset \mathcal{C}_1$, and deduce from Lemma 4.4.5 that $v_1(\underline{\mathrm{x}}) = \dfrac{\mu_1}{\rho}$.

3. We now introduce

$$\bar{x} = \inf\{y \geq \underline{\mathrm{x}} \,|\, y \in \mathcal{S}_1\}.$$

Observe that $\bar{x} + (1-\lambda)g \in \mathcal{D}_0$. Moreover, according to Lemma 4.4.3, $\bar{x} + (1-\lambda)g \notin \mathcal{S}_0$ and thus a left neighborhood of $\bar{x} + (1-\lambda)g$ belongs to $\mathcal{C}_0$. We first prove that $\bar{x} + (1-\lambda)g$ cannot be a left boundary of $\mathcal{D}_0$. On the contrary, we would have from Lemma 4.4.5,

$$v_1(\bar{x}) \quad = \quad v_0(\bar{x} + (1-\lambda)g) \;=\; \frac{\mu_0}{\rho} \;<\; \frac{\mu_1}{\rho} \;=\; v_1(\underline{\mathrm{x}}),$$

which contradicts the fact that v_1 is increasing. Therefore, $\bar{x} + (1-\lambda)g \in \overset{o}{\mathcal{D}}_0$, and we can find $y < \bar{x}$ such that $y + (1-\lambda)g$ is a left boundary of $\mathcal{D}_0$. Hence,

$$v_1(\bar{x}) \quad = \quad v_0(\bar{x} + (1-\lambda)g) \;=\; \bar{x} - y + v_0(y + (1-\lambda)g) \;\leq\; \bar{x} - y + v_1(y).$$

Since the reverse inequality is always true, we obtain that $y \in \mathcal{S}_1$ which contradicts the definition of $\bar{x}$. We conclude that $\bar{x}$ cannot be strictly positive, which is a contradiction with the first step. This proves finally that $\mathcal{S}_1^*$ is empty, i.e. $\mathcal{S}_1 = \{0\}$.

(ii) Assume that $v_0((1-\lambda)g) \geq \dfrac{\mu_1}{\rho}$. Let y be a left boundary of $\mathcal{D}_1$. We shall prove that y necessarily equals zero. If not, according to Lemma 4.4.5, $v_1(y) \leq \dfrac{\mu_1}{\rho} \leq v_1(0)$ where the second inequality comes from the hypothesis and (4.3.9). Since the function v_1 is strictly increasing, we get the desired contradiction. Therefore, $\mathcal{D}_1 = [0, a]$. It remains to prove that a is infinite. From Lemma 4.4.4, the open set (a, ∞) belongs to $\mathcal{C}_1$ if $a < \infty$. Using the regularity of v_1 on $\mathcal{C}_1$, we get by the same reasoning as in the proof of Lemma 4.4.5 that $v_1(a) = \dfrac{\mu_1}{\rho}$, which gives the same contradiction as before. Hence, $\mathcal{D}_1 = [0, \infty)$. We then have for any $x > 0$,

$$v_1(x) \quad = \quad x + v_0((1-\lambda)g) \;\leq\; v_0(x + (1-\lambda)g).$$

Since the reverse inequality is always true by definition, we conclude that $\mathcal{S}_1 = [0, \infty)$. □

The next proposition describes the structure of the switching region from technology $i = 0$ to $i = 1$, in the case where the growth rate μ_1 in the modern technology $i = 1$, is large enough.

Proposition 4.4.2 *Suppose that*

$$\frac{\mu_1 - \mu_0}{\rho} \;>\; \hat{x}_1 + g - \hat{x}_0, \quad \textit{and} \quad \hat{V}_0((1-\lambda)g) \;<\; \frac{\mu_1}{\rho}.$$

Then, there exists $x_{01}^* \in [g, \infty)$ *s.t.*

$$\mathcal{S}_0^* \;=\; [x_{01}^*, \infty).$$

Proof. We first notice that $\mathcal{S}_0^* \neq \emptyset$. On the contrary, we would have $v_0 = \hat{V}_0$, and so $\hat{V}_0(x) \geq v_1(x-g) \geq \hat{V}_1(x-g)$ for all x, which is in contradiction with Lemma 4.4.2. Moreover, since $v_1(x-g) = v_0(x) > 0$ for all $x \in \mathcal{S}_0^*$, we deduce that $\mathcal{S}_0^* \subset [g, \infty)$ and so

$$x_{01}^* \;:=\; \inf \mathcal{S}_0^* \;\in\; [g, \infty).$$

Let us now consider the function

$$w_0(x) \;=\; \begin{cases} v_0(x), & x < x_{01}^* \\ v_1(x-g), & x \geq x_{01}^*. \end{cases}$$

We claim that w_0 is a viscosity solution, with linear growth condition and boundary data $w_0(0^+) = 0$, to

$$\min\left[\rho w_0(x) - \mathcal{L}_0 w_0(x), w_0'(x) - 1, w_0(x) - v_1(x-g)\right] \;=\; 0, \;\; x > 0.$$

For $x < x_{01}^*$, this is clear since $w_0 = v_0$ on $(0, x_{01}^*)$. For $x > x_{01}^*$, we see that $w_0' \geq 1$ and

$$\begin{aligned} \rho w_0 - \mathcal{L}_0 w_0 \;&=\; (\rho v_1 - \mathcal{L}_1 v_1 + (\mu_1 - \mu_0) v_1')(x-g) \\ &\geq\; (\mu_1 - \mu_0) v_1'(x-g) \;\geq\; 0. \end{aligned}$$

Hence, the viscosity property is also satisfied for $x > x_{01}^*$. It remains to check the viscosity property for $x = x_{01}^*$. The viscosity subsolution property at x_{01}^* is trivial since $w_0(x_{01}^*) = v_1(x_{01}^* - g)$. For the viscosity supersolution property, take some C^2 test function φ s.t. x_{01}^* is a local minimum of $w_0 - \varphi$. From the smooth-fit condition of the value function v_0 at the switching boundary, it follows that w_0 is C^1 at x_{01}^*. Hence $w_0'(x_{01}^*) = \varphi'(x_{01}^*)$. Moreover, since $w_0 = v_0$ is C^2 for $x < x_{01}^*$, we also have $\varphi''(x_{01}^*) \leq w_0''(x_{01}^{*-}) := \lim_{x \nearrow x_{01}^*} w''(x)$. Since $\rho w_0(x) - \mathcal{L}_0 w_0(x) \geq 0$ for $x < x_{01}^*$, we deduce by sending x to x_{01}^* :

$$\rho w_0(x_{01}^*) - \mathcal{L}_0 \varphi(x_{01}^*) \;\;\geq\;\; 0.$$

This implies the required viscosity supersolution inequality at $x = x_{01}^*$. By uniqueness, we conclude that $w_0 = v_0$, which proves that $\mathcal{S}_0^* = [x_{01}^*, \infty)$. □

4.5 Main result and description of the solution

We give an explicit description of the structure of the solution to our control problem, which depends crucially on parameter values.

4.5.1 The case : $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$

Theorem 4.5.1 *Suppose that $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$. Then, we have $v_0(x) = \hat{V}_0(x)$ and $v_1(x) = \hat{V}_0(x+(1-\lambda)g) = x+(1-\lambda)g - x_0 + \frac{\mu_0}{\rho}$. It is optimal to never switch from regime 0 to regime 1. In regime 1, it is optimal to distribute all the surplus as dividends and to switch to regime 0.*

Proof. Under the condition of the theorem, and since $v_0 \geq \hat{V}_0$, we have $v_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$. By Proposition 4.4.1, this implies $\mathcal{S}_1 = \mathcal{D}_1 = \mathbb{R}_+$. Recalling also the boundary data $v_1(0) = v_0((1-\lambda)g)$, we get $v_1(x) = x + v_0((1-\lambda)g)$ for $x \geq 0$. We next prove that the region $\mathcal{S}_0^*$ is empty. To see this, we have to prove that for $x \geq g$, $v_0(x) > v_1(x-g)$. Let us consider for $x \geq g$ the function $\theta(x) = v_0(x) - (x-g+v_0((1-\lambda)g))$. Since $\lambda > 0$, we have $\theta(g) > 0$. Moreover, $\theta'(x) = v_0'(x) - 1 \geq 0$. Thus, $\theta(x) > 0$ for $x \geq g$ which is equivalent to $\mathcal{S}_0^* = \emptyset$. As a consequence, v_0 is a smooth solution of the variational inequality

$$\min\left[\rho v(x) - \mathcal{L}_0 v(x), v'(x) - 1\right] = 0,$$

with initial condition $v(0) = 0$. By uniqueness, we deduce that $v_0 = \hat{V}_0$. To close the proof, it suffices to note that $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$ implies that $(1-\lambda)g \geq \hat{x}_0$. Therefore, $v_0((1-\lambda)g) = (1-\lambda)g - x_0 + \frac{\mu_0}{\rho}$. □

4.5.2 The case : $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$

First observe that in this case, we have

$$v_0((1-\lambda)g) \quad < \quad \frac{\mu_1}{\rho}.$$

Indeed, on the contrary, from Theorem 4.5.1, we would get $v_0 = \hat{V}_0$, and so an obvious contradiction $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$ with the considered case. From Proposition 4.4.1, we then have $\mathcal{S}_1 = \{0\}$ so that v_1 is the unique viscosity solution to

$$\min\left[\rho v_1 - \mathcal{L}_1 v_1, v_1' - 1\right] \quad = \quad 0, \quad x > 0,$$

with the boundary data $v_1(0) = v_0((1-\lambda)g)$. Therefore, v_1 is the firm value problem in technology $i = 1$ with liquidation value $v_0((1-\lambda)g)$:

$$v_1(x) \quad = \quad \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} v_0((1-\lambda)g)\right], \tag{4.5.1}$$

The form of v_1 is described in (4.4.3) with liquidation value $L = v_0((1-\lambda)g)$: we denote $x_1 = x_1^L$ the corresponding threshold.

Remark 4.5.1 Since v_1 and $\hat{V}_1$ are increasing with $v_1(x_1) = \hat{V}_1(\hat{x}_1) = \frac{\mu_1}{\rho}$, we have $x_1 \leq \hat{x}_1$.

Notice that the expression of v_1 is not completely explicit since we do not know at this stage the liquidation value $v_0((1-\lambda)g)$. The next result give an explicit solution when

$$\frac{\mu_1 - \mu_0}{\rho} \quad \leq \quad \hat{x}_1 + g - \hat{x}_0.$$

Theorem 4.5.2 *Suppose that*

$$\hat{V}_0((1-\lambda)g) \ < \ \frac{\mu_1}{\rho} \quad \leq \quad \frac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0. \tag{4.5.2}$$

Then $v_0 = \hat{V}_0$ and $v_1 = \hat{V}_1$. It is never optimal, once in regime $i = 0$, to switch to regime $i = 1$. In regime 1, it is optimal to switch to regime 0 at the threshold $x = 0$.

Proof. From Lemma 4.4.1 and Lemma 4.4.2, and recalling the variational inequalities (4.4.2) and (4.4.4), we see that $\hat{V}_0$ and $\hat{V}_1$ are viscosity solutions to

$$\begin{aligned} \min\Big[\rho\hat{V}_0(x) - \mathcal{L}_0\hat{V}_0(x)\ ,\ \hat{V}_0'(x) - 1\ ,\ \hat{V}_0(x) - \hat{V}_1(x-g)\Big] &= 0, \quad x > 0, \\ \min\Big[\rho\hat{V}_1(x) - \mathcal{L}_1\hat{V}_1(x)\ ,\ \hat{V}_1'(x) - 1\ ,\ \hat{V}_1(x) - \hat{V}_0(x + (1-\lambda)g)\Big] &= 0, \quad x > 0, \end{aligned}$$

together with the boundary data $V_0(0^+) = 0$ and $\hat{V}_1(0^+) = \hat{V}_0((1-\lambda)g)$. By uniqueness to this system of variational inequalities, we conclude that $(v_0, v_1) = (\hat{V}_0, \hat{V}_1)$. □

In the sequel, we suppose that

$$\frac{\mu_1 - \mu_0}{\rho} \quad > \quad \hat{x}_1 + g - \hat{x}_0. \tag{4.5.3}$$

From Proposition 4.4.2, the switching region from regime 0 to regime 1 has the form :

$$\mathcal{S}_0^* \quad = \quad \{x > 0\ : v_0(x) \ = \ v_1(x-g)\} \ = \ [x_{01}^*, \infty),$$

for some $x_{01}^* \in [g, \infty)$. Moreover, since $x_1 \leq \hat{x}_1$ (see Remark 4.5.1), the above condition (4.5.3) implies $\frac{\mu_1 - \mu_0}{\rho} > x_1 + g - \hat{x}_0$. By same arguments as in Remark 4.4.3, there exists some $\bar{x}_{01} \geq g$ s.t.

$$\max\Big(\hat{V}_0(x), v_1(x-g)\Big) \quad = \quad \begin{cases} \hat{V}_0(x), & x \leq \bar{x}_{01} \\ v_1(x-g), & x > \bar{x}_{01} \end{cases}$$

Following [24], we introduce the pure stopping time problem

$$\bar{v}_0(x) \quad = \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \max\Big(\hat{V}_0(R^{x,0}_{\tau\wedge T_0}), v_1(R^{x,0}_{\tau \wedge T_0} - g)\Big)\right], \tag{4.5.4}$$

where $\mathcal{T}$ denotes the set of stopping times valued in $[0, \infty]$. We also denote $\mathcal{E}_0$ the exercice region for $\bar{v}_0$:

$$\mathcal{E}_0 \quad = \quad \Big\{x \geq 0\ : \bar{v}_0(x) = \max\Big(\hat{V}_0(x), v_1(x-g)\Big)\Big\}.$$

The next result shows that the original mixed singular/switching control problems may be reformulated as a coupled pure optimal stopping time and pure singular problem.

Theorem 4.5.3 *Suppose that*

$$\hat{V}_0((1-\lambda)g) \;<\; \frac{\mu_1}{\rho} \quad \textit{and} \quad \frac{\mu_1-\mu_0}{\rho} \;>\; \hat{x}_1+g-\hat{x}_0. \tag{4.5.5}$$

Then, we have

$$v_0 \;=\; \bar{v}_0$$

and v_1 *given by* (4.5.1). *Moreover,*

$$\mathcal{E}_0 \;=\; \left\{0 \le x < \bar{x}_{01} \;: v_0(x) = \hat{V}_0(x)\right\} \cup [x^*_{01}, \infty).$$

Proof. The proof follows along the lines of those of Theorem 3.1 in [24]. We will give only the road map of it in our context and omit the details.
Let us first note that the process $(e^{-\rho(t\wedge T_0)}v_0(R^{x,0}_{t\wedge T_0}))_{t\geq 0}$ is a supermartingale that dominates the function $\max(\hat{V}_0, v_1(.-g))$. Therefore, according to Snell envelope theory, we have $v_0 \ge \bar{v}_0$.
To prove the reverse inequality, it is enough to show that $\bar{v}_0' \ge 1$ (see Proposition 3.4 in [24]) and to use the uniqueness result of Theorem 4.3.1. To this end, we will precise the shape of the exercise region $\mathcal{E}_0$. According to Lemma 4.3 by Villeneuve [66], $\bar{x}_{01}$ does not belong to $\mathcal{E}_0$. Thus, the exercise region can be decomposed into two subregions

$$\mathcal{E}_{00} = \left\{x < \bar{x}_{01} : v_0(x) = \hat{V}_0(x)\right\}$$

and

$$\mathcal{E}_{01} = \left\{x > \bar{x}_{01} : v_0(x) = v_1(x-g)\right\}.$$

Two cases have to be considered :

Case (i). If the subregion $\mathcal{E}_{00}$ is empty, the optimal stopping problem defined by $\bar{v}_0$ can be solved explicitly, and we have, see [24] Lemma 3.3,

$$\bar{v}_0 = \begin{cases} \frac{e^{m_0^+ x}-e^{m_0^- x}}{e^{m_0^+ x^*_{01}}-e^{m_0^- x^*_{01}}} v_1(x^*_{01}-g) & x < x^*_{01} \\ v_1(x-g) & x \ge x^*_{01}. \end{cases}$$

The smooth-fit principle allows us to conclude that $\bar{v}_0' \ge 1$ since $v_1' \ge 1$.

Case (ii). If the subregion $\mathcal{E}_{00}$ is non empty, we can prove using the arguments of Proposition 3.5 and Lemma 3.4 in [24] that

$$\mathcal{E}_0 = [0,a] \cup [x^*_{01}, \infty),$$

with $a \ge \hat{x}_0$ and the value function $\bar{v}_0$ satisfies

$$\bar{v}_0(x) = Ae^{m_0^+ x} + Be^{m_0^- x} \text{ for } x \in (a, x^*_{01}).$$

The smooth fit principle gives $\bar{v}_0'(a) = \hat{V}_0'(a) \geq 1$ and $\bar{v}_0'(x_{01}^*) = v_1'(x_{01}^* - g) \geq 1$. Clearly, $\bar{v}_0$ is convex in a right neighborhood of a since $\hat{V}_0$ is linear at a. Therefore, if $\bar{v}_0$ remains convex on (a, x_{01}^*), the proof is over. If not, the second derivative of $\bar{v}_0$ given by $A(m_0^+)^2 e^{m_0^+ x} + B(m_0^-)^2 e^{m_0^- x}$ vanishes at most one time on (a, x_{01}^*), say in d. Hence,

$$1 = \bar{v}_0'(a) \leq (\bar{v}_0)'(x) \leq \bar{v}_0'(d) \text{ for } x \in (a, d),$$

and

$$1 \leq \bar{v}_0'(x_{01}^*) \leq \bar{v}_0'(x) \leq \bar{v}_0'(d) \text{ for } x \in (d, x_{01}^*),$$

which completes the proof. $\Box$

Notice that the representation (4.5.1)-(4.5.4) of pure optimal singular and stopping problems for v_1 and v_0 is coupled, and so not easily computable. We decouple this representation by considering the sequence of pure optimal stopping and singular control problems, starting from $\hat{V}_1^{(0)} = \hat{V}_1$ and $\hat{V}_0^{(0)} = \hat{V}_0$:

$$\begin{aligned}
\hat{V}_0^{(k)}(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \max\left(\hat{V}_0(R^{x,0}_{\tau \wedge T_0}), \hat{V}_1^{(k-1)}(R^{x,0}_{\tau \wedge T_0} - g)\right)\right], \quad k \geq 1, \\
\hat{V}_1^{(k)}(x) &= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} \hat{V}_0^{(k)}((1-\lambda)g)\right], \quad k \geq 1.
\end{aligned}$$

The next result shows the convergence of this procedure.

Proposition 4.5.1 *Under the conditions* (4.5.5) *of Theorem 4.5.3, we have for all $x > 0$:*

$$\lim_{k \to \infty} \hat{V}_0^{(k)}(x) = v_0(x), \qquad \lim_{k \to \infty} \hat{V}_1^{(k)}(x) = v_1(x).$$

Proof. We will first prove that the increasing sequence $(\hat{V}_0^{(k)}, \hat{V}_1^{(k)})$ converges uniformly on every compact subsets of $\mathbb{R}_+$. To see this, we will apply Arzela-Ascoli Theorem by first proving the equi-continuity of the functions $\hat{V}_i^{(k)}$. Let us first remark that the functions $\hat{V}_1^{(k)}$ are Lipschitz continuous uniformly in k since they are concave with bounded first derivative (see Remark 4.4.2) independently of k. Let us also check that the functions $\hat{V}_0^{(k)}$ are Lipschitz continuous uniformly in k. Using the inequality $\max(a, b) - \max(c, d) \leq \max(a - c, b - d)$, and by setting

$$\Delta(x, y) = \max\left(\hat{V}_0(R^{x,0}_{\tau \wedge T_0}) - \hat{V}_0(R^{y,0}_{\tau \wedge T_0}), \hat{V}_1^{(k-1)}(R^{x,0}_{\tau \wedge T_0} - g) - \hat{V}_1^{(k-1)}(R^{y,0}_{\tau \wedge T_0} - g)\right),$$

we get by recalling also that $\hat{V}_0$ is Lipschitz (see Remark 4.4.2) :

$$\begin{aligned}
|\hat{V}_0^{(k)}(x) - \hat{V}_1^{(k)}(y)| &\leq \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} |\Delta(x, y)|\right] \\
&\leq K_0 \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} |R^{x,0}_{\tau \wedge T_0} - R^{y,0}_{\tau \wedge T_0}|\right] \\
&\leq K_0 |x - y| \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} |\mu_0 \tau \wedge T_0 + \sigma W_{\tau \wedge T_0}|\right] \\
&\leq K_1 |x - y|.
\end{aligned}$$

According to Corollary 4.3.7, the set $\{(\hat{V}_0^{(k)}(x), \hat{V}_1^{(k)}(x)), k \in \mathbb{N}\}$ is bounded for every $x > 0$. Therefore, Arzela-Ascoli Theorem gives that the increasing sequence $(\hat{V}_0^{(k)}, \hat{V}_1^{(k)})$ converges uniformly on every compact subset of $\mathbb{R}_+$ to some $(\hat{V}_0^{(\infty)}, \hat{V}_1^{(\infty)})$.
On the other hand, for a fixed k, $(\hat{V}_0^{(k)}, \hat{V}_1^{(k)})$ is the unique viscosity solution with linear growth to the system of variational inequalities

$$F_0^{(k)}(u_0, u_0', u_0'') = \min\Big(\rho u_0 - \mathcal{L}_0 u_0, u_0 - \max(\hat{V}_0, \hat{V}_1^{(k-1)}(. - g))\Big) \; = \; 0,$$

$$F_1(u_1, u_1', u_1'') = \min\big(\rho u_1 - \mathcal{L}_1 u_1, u_1' - 1\big) \; = \; 0,$$

with initial condition $u_0(0) = 0$, $u_1(0) = \hat{V}_0^{(k)}((1-\lambda)g)$.
Since $\hat{V}_1^{(k-1)}$ converges uniformly on every compact subset of $\mathbb{R}_+$, the Hamiltonian $F_0^{(k)}$ converges to F_0 on every compact subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, with

$$F_0(u, u', u'') = \min\Big(\rho u - \mathcal{L}_0 u, u - \max(\hat{V}_0, \hat{V}_1^{\infty}(. - g))\Big) \; = \; 0.$$

According to standard stability results for viscosity solution, see for instance Lemma 6.2 page 73 in Fleming and Soner [28], the couple $(\hat{V}_0^{(\infty)}, \hat{V}_1^{(\infty)})$ is a viscosity solution of the system of variational inequalities

$$\min\Big(\rho\hat{V}_0^{\infty} - \mathcal{L}_0\hat{V}_0^{\infty}, \hat{V}_0^{\infty} - \max(\hat{V}_0, \hat{V}_1^{\infty}(. - g))\Big) \; = \; 0, \tag{4.5.6}$$

$$\min\Big(\rho\hat{V}_1^{\infty} - \mathcal{L}_1\hat{V}_1^{\infty}, (\hat{V}_1^{\infty})' - 1\Big) \; = \; 0, \tag{4.5.7}$$

with initial conditions $\hat{V}_1^{\infty}(0) = \hat{V}_0^{\infty}((1-\lambda)g)$ and $\hat{V}_0^{\infty}(0) = 0$. By uniqueness to the system (4.5.6)-(4.5.7), we conclude that $\hat{V}_0^{\infty} = \bar{v}_0 = v_0$ and $\hat{V}_1^{\infty} = v_1$. □

We will close this section by describing the optimal strategy. According to Proposition 4.5.1, the value functions can be constructed recursively starting from $(\hat{V}_0, \hat{V}_1)$. Two cases have to be considered :

Case A : $\hat{V}_0^{(1)}((1-\lambda)g) = \hat{V}_0((1-\lambda)g)$. Then we have

$$\begin{aligned}
\hat{V}_1^{(1)}(x) &= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1}\hat{V}_0^{(1)}((1-\lambda)g)\right] \\
&= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1}\hat{V}_0((1-\lambda)g)\right] \\
&= \hat{V}_1(x).
\end{aligned}$$

Therefore, we deduce by a straightforward induction that the sequence $(\hat{V}_0^{(k)})_k$ is constant for $k \geq 1$ and the sequence $(\hat{V}_1^{(k)})_k$ is constant for $k \geq 0$. Therefore, we deduce from Proposition 4.5.1 that $v_0 = \hat{V}_0^{(1)}$ and $v_1 = \hat{V}_1$.

In regime 0, the optimal strategy consists in computing the optimal thresholds a and x_{01}^* associated to the optimal stopping problem $\hat{V}_0^{(1)}$. It is optimal to switch from regime 0 to regime 1 if the state process R^0 crosses the threshold x_{01}^* while it is optimal to pay dividends and therefore abandon the growth opportunity forever if R^0 falls below the threshold a. At the level a, it is too costly to wait reaching the threshold x_{01}^* even if the growth option is valuable. The shareholders prefer to receive today dividends than waiting a more profitable payment in the future.

In regime 1, the optimal strategy consists in paying dividends above $\hat{x}_1$ and switching to regime 0 only when the manager is being forced by its cash constraints.

Case B : $\hat{V}_0^{(1)}((1-\lambda)g) > \hat{V}_0((1-\lambda)g)$. Let us introduce the sequence

$$\begin{aligned}
\hat{\theta}_0^{(k)}(x) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau\wedge T_0)}\hat{\theta}_1^{(k-1)}(R^{x,0}_{\tau\wedge T_0}-g)\right], \quad k\geq 1,\\
\hat{\theta}_1^{(k)}(x) &= \sup_{Z\in\mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t}dZ_t + e^{-\rho T_1}\hat{\theta}_0^{(k)}((1-\lambda)g)\right], \quad k\geq 1.
\end{aligned}$$

starting from $\hat{\theta}_1^{(0)} = \hat{V}_1$ and $\hat{\theta}_0^{(0)} = \hat{V}_0$. Proceeding analogously as in the proof of Proposition 4.5.1, we can prove that the sequence $(\theta_0^{(k)}, \theta_1^{(k)})$ converges to $(\theta_0^{(\infty)}, \theta_1^{(\infty)})$ solution of the system of variational inequalities :

$$\min\left(\rho\hat{\theta}_0^{\infty} - \mathcal{L}_0\hat{\theta}_0^{\infty}, \hat{\theta}_0^{\infty} - \hat{\theta}_1^{\infty}(.-g)\right) = 0,$$

$$\min\left(\rho\hat{\theta}_1^{\infty} - \mathcal{L}_1\hat{\theta}_1^{\infty}, \hat{(\theta_1^{\infty})}' - 1\right) = 0,$$

with initial conditions $\hat{\theta}_1^{\infty}(0) = \hat{\theta}_0^{\infty}((1-\lambda)g)$ and $\hat{\theta}_0^{\infty}(0) = 0$.
Note that the function $\hat{\theta}_0^{\infty}$ corresponds to the managerial decision to accumulate cash reserve at the expense of shareholder's dividend payment in order to invest in the modern technology.

The key feature of our model in **case B**, which has to be viewed as the analogue of Proposition 3.5 in [24], can be summarized as follows :

⋆ If the net expected value evaluated at the threshold $\hat{x}_0$ dominates the firm value running under the old technology that is $\hat{\theta}_0^{\infty}(\hat{x}_0) > \hat{V}_0(\hat{x}_0)$ then the manager postpones dividend distribution in order to invest in the modern technology and thus $v_0 = \hat{\theta}_0^{\infty}$. Moreover, in regime 1, the manager always prefers to run under the modern technology until the cash process X_t^1 reaches zero forcing the manager to return back in regime 0 with the value $\hat{\theta}_0^{\infty}((1-\lambda)g)$, that is $v_1 = \hat{\theta}_1^{\infty}$.

⋆ If, on the contrary $\hat{\theta}_0^{\infty}(\hat{x}_0) \leq \hat{V}_0(\hat{x}_0)$ then the manager optimally ignores the strategy $\hat{\theta}_0^{\infty}$. Several situations can occur. For small values of the cash process ($X_t^0 \leq a$), the manager optimally runs the firm under the old technology and pays out any surplus above $\hat{x}_0$ as dividends. For high values of the cash process ($X_t^0 \geq x_{01}^*$), the manager switches optimally to regime one. For intermediary values of the cash process ($a \leq X_t^0 \leq x_{01}^*$), there is an inaction region where the manager has not enough information to decide whether or not the investment is valuable.

We summarize all the results in Synthetic Table 1 and Figure 1.

Synthetic table 1

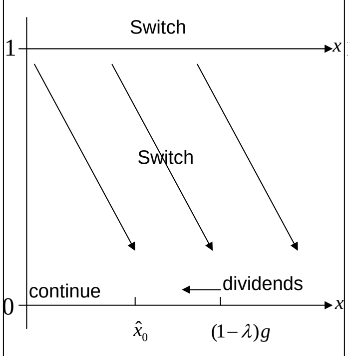
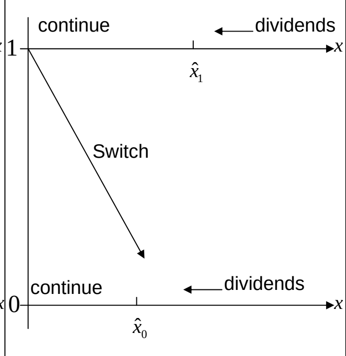
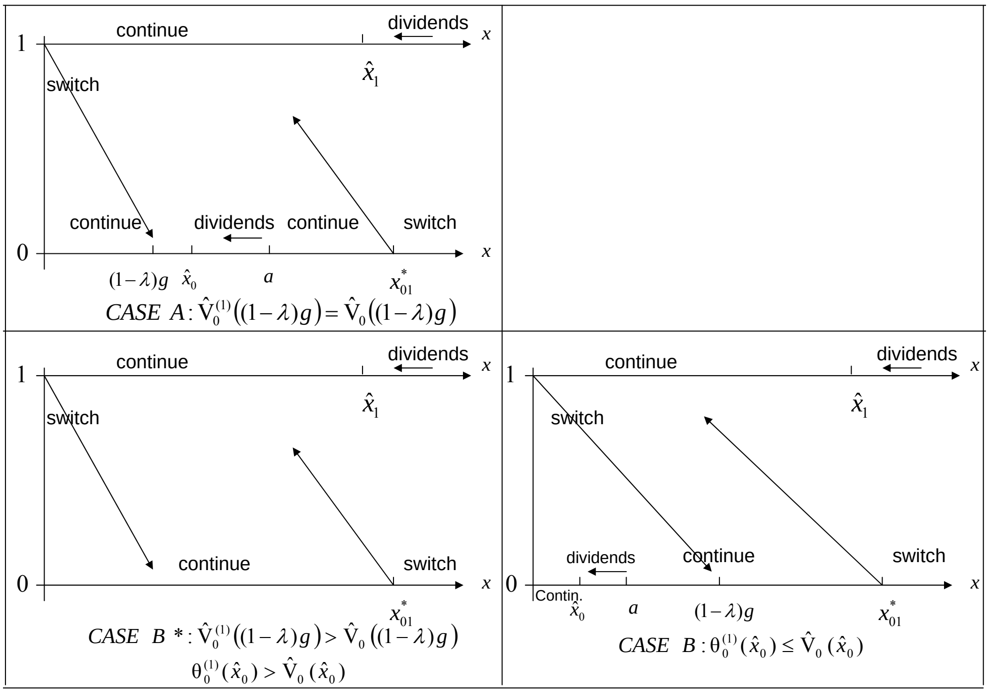
$\frac{\mu_1}{\rho} \le \hat{V}_0((1-\lambda)g)$	$\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho} \le \frac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0$	$\frac{\mu_1}{\rho} > \max\left(\hat{V}_0((1-\lambda)g), \frac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0\right)$
$v_0(x) = \hat{V}_0(x)$ $v_1(x) = x + (1-\lambda)g - \hat{x}_0 + \frac{\mu_0}{\rho}$	$v_0(x) = \hat{V}_0(x)$ $v_1(x) = \hat{V}_1(x)$	$v_0(x) = \hat{V}_0^{\infty}(x)$ $v_1(x) = \hat{V}_1^{\infty}(x)$
		See figure 1

Figure 1



Appendix A : Proof of Theorem 4.3.1

We divide the proof in several steps.

Proof of the continuity of v_1 on $(0, \infty)$.
We prove that v_1 is continuous at any $y > 0$. We fix an arbitrary small $\varepsilon > 0$. Applying the dynamic programming principle **(DP)** to v_1, there exists a control $\alpha = (Z, (\tau_n)_{n\geq 1}) \in \mathcal{A}$ s.t.

$$\begin{aligned} v_1(y) - \frac{\varepsilon}{3} &\leq \mathbb{E}\left[\int_0^{(\tau_1\wedge T)^-} e^{-\rho t} dZ_t + e^{-\rho(\tau_1\wedge T)}\left(v_1(X_T^{y,1})1_{T<\tau_1} + v_0(X_{\tau_1}^{y,1})1_{\tau_1\leq T}\right)\right], \\ &= \mathbb{E}\left[\int_0^{(\tau_1\wedge T)^-} e^{-\rho t} dZ_t + e^{-\rho(\tau_1\wedge T)} v_0(X_{\tau_1}^{y,1})1_{\tau_1\leq T}\right], \end{aligned} \quad (4.A.1)$$

with $T = T^{y,1,\alpha}$ the bankruptcy time of the process $X^{y,1,\alpha}$, and since $v_1(X_T^{y,1}) = 0$ for $X_T^{y,1} < 0$.

For any $0 < x < y$, let $\theta = T^{y-x,1,\alpha}$ be the bankruptcy time of the process $X^{y-x,1,\alpha}$. We notice that $\theta \leq T$ and $X^{y-x,1,\alpha} = X^{y,1,\alpha} - x$ for all $0 < t < \theta \leq T$. Applying the dynamic programming principle **(DP)**, we then have

$$\begin{aligned} v_1(y-x) &\geq \mathbb{E}\left[\int_0^{(\theta\wedge\tau_1)^-} e^{-\rho t} dZ_t + e^{-\rho(\theta\wedge\tau_1)}\left(v_1(X_\theta^{y-x,1})1_{\theta<\tau_1} + v_0(X_{\tau_1}^{y-x,1})1_{\tau_1\leq\theta}\right)\right] \\ &\geq \mathbb{E}\left[\int_0^{(\theta\wedge\tau_1)^-} e^{-\rho t} dZ_t + e^{-\rho(\theta\wedge\tau_1)} v_0(X_{\tau_1}^{y-x,1})1_{\tau_1\leq\theta}\right] \\ &\geq \mathbb{E}\left[\int_0^{(\tau_1\wedge T)^-} e^{-\rho t} dZ_t + e^{-\rho(\tau_1\wedge T)} v_0(X_{\tau_1}^{y,1})1_{\tau_1\leq T}\right] - \mathbb{E}\left[\int_{\theta\wedge\tau_1}^{(T\wedge\tau_1)^-} e^{-\rho t} dZ_t\right] \\ &\quad + \mathbb{E}\left[e^{-\rho(\theta\wedge\tau_1)} v_0(X_{\tau_1}^{y-x,1})1_{\tau_1\leq\theta} - e^{-\rho(T\wedge\tau_1)} v_0(X_{\tau_1}^{y,1})1_{\tau_1\leq T}\right] \end{aligned} \quad (4.A.2)$$

Notice that $\theta \to T$ as x goes to zero. Hence, by the continuity of v_0 and the dominated convergence theorem, one can find $0 < \delta_1 < y$ s.t. for $0 < x < \delta_1$:

$$\mathbb{E}\left[e^{-\rho(\theta\wedge\tau_1)} v_0(X_{\tau_1}^{y-x,1})1_{\tau_1\leq\theta} - e^{-\rho(T\wedge\tau_1)} v_0(X_{\tau_1}^{y,1})1_{\tau_1\leq T}\right] \geq -\frac{\varepsilon}{3}. \quad (4.A.3)$$

We also have

$$-\mathbb{E}\left[\int_{\theta\wedge\tau_1}^{(T\wedge\tau_1)^-} e^{-\rho t} dZ_t\right] \geq -\mathbb{E}\left[Z_{(T\wedge\tau_1)^-} - Z_{\theta\wedge\tau_1}\right]$$

From the dominated convergence theorem, one can find $0 < \delta_2 < y$ s.t. for $0 < x < \delta_2$:

$$-\mathbb{E}\left[\int_{\theta\wedge\tau_1}^{(T\wedge\tau_1)^-} e^{-\rho t} dZ_t\right] \geq -\frac{\varepsilon}{3} \quad (4.A.4)$$

Plugging inequalities (4.A.3) and (4.A.4) into (4.A.2), we obtain for $0 < x < \min\{\delta_1, \delta_2\}$

$$v_1(y-x) \geq \mathbb{E}\left[\int_0^{(\tau_1\wedge T)^-} e^{-\rho t} dZ_t + e^{-\rho(\tau_1\wedge T)} v_0(X_{\tau_1}^{y,1}) 1_{\tau_1 \leq T}\right] - \frac{2\varepsilon}{3}$$

Using the inequality (4.A.1), and recalling that v_1 is nondecreasing, this implies

$$0 \leq v_1(y) - v_1(y-x) \leq \varepsilon,$$

which shows the left-continuity of v_1. By proceeding exactly in the same manner, we may obtain for a given $y > 0$ and any arbitrary $\varepsilon > 0$, the existence of $0 < \delta < y$ such that for all $0 < x < \delta$,

$$0 \leq v_1(y+x) - v_1(y) \leq \varepsilon$$

which shows the right-continuity of v_1. □

Proof of supersolution property.

Fix $i \in \{0,1\}$. Consider any $\overline{x} \in (0,\infty)$ and $\varphi \in C^2(0,\infty)$ s.t. $\overline{x}$ is a minimum of $v_i - \varphi$ in a neighborhood $B_\varepsilon(\overline{x}) = (\overline{x}-\varepsilon, \overline{x}+\varepsilon)$ of $\overline{x}$, $\overline{x} > \varepsilon > 0$, and $v_i(\overline{x}) = \varphi(\overline{x})$. First, by considering the admissible control $\overline{\alpha} = (\overline{Z}, \overline{\tau}_n, n \geq 1)$ where we decide to take immediate switching control, i.e. $\overline{\tau}_1 = 0$, while deciding not to distribute any dividend $\overline{Z} = 0$, we obtain

$$v_i(\overline{x}) \geq v_{i-1}(\overline{x} - g_{i,1-1}). \tag{4.A.5}$$

On the other hand, let us consider the admissible control $\hat{\alpha} = (\hat{Z}, \hat{\tau}_n, n \geq 1)$ where we decide to never switch regime, while the dividend policy is defined by $\hat{Z}_t = \eta$ for $t \geq 0$, with $0 \leq \eta \leq \varepsilon$. Define the exit time $\tau_\varepsilon = \inf\{t \geq 0, X_t^{\bar{x},i} \notin \overline{B}_\varepsilon(\overline{x})\}$. We notice that $\tau_\varepsilon < T$. From the dynamic programming principle **(DP)**, we have

$$\begin{aligned} \varphi(\overline{x}) = v(\overline{x}) &\geq \mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-\rho t} d\hat{Z}_t + e^{-\rho(\tau_\varepsilon\wedge h)} v_i(X_{\tau_\varepsilon\wedge h}^{\bar{x},i})\right] \\ &\geq \mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-\rho t} d\hat{Z}_t + e^{-\rho(\tau_\varepsilon\wedge h)} \varphi(X_{\tau_\varepsilon\wedge h}^{\bar{x},i})\right]. \end{aligned} \tag{4.A.6}$$

Applying Itô's formula to the process $e^{-\rho t}\varphi(X_t^{\bar{x},i})$ between 0 and $\tau_\varepsilon \wedge h$, and taking the expectation, we obtain

$$\begin{aligned} \mathbb{E}\left[e^{-\rho(\tau_\varepsilon\wedge h)} \varphi(X_{\tau_\varepsilon\wedge h}^{\bar{x},i})\right] &= \varphi(\overline{x}) + \mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-\rho t}(-\rho\varphi + \mathcal{L}_i\varphi)(X_t^{\bar{x},i}) dt\right] \\ &\quad + \mathbb{E}\left[\sum_{0\leq t\leq \tau_\varepsilon\wedge h} e^{-\rho t}[\varphi(X_t^{\bar{x},i}) - \varphi(X_{t^-}^{\bar{x},i})]\right]. \end{aligned} \tag{4.A.7}$$

Combining relations (4.A.6) and (4.A.7), we have

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-\rho t}(\rho\varphi-\mathcal{L}_i\varphi)(X_t^{\bar{x},i})dt\right]-\mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-\rho t}d\hat{Z}_t\right]$$
$$-\mathbb{E}\left[\sum_{0\le t\le\tau_\varepsilon\wedge h} e^{-\rho t}[\varphi(X_t^{\bar{x},i})-\varphi(X_{t^-}^{\bar{x},i})]\right] \ \ge\ 0. \qquad (4.A.8)$$

$\star$ Take first $\eta = 0$. We then observe that X is continuous on $[0, \tau_\varepsilon \wedge h]$ and only the first term of the relation (4.A.8) is non zero. By dividing the above inequality by h with $h \to 0$, we conclude that

$$(\rho\varphi-\mathcal{L}_i\varphi)(\overline{x}) \ge 0. \qquad (4.A.9)$$

$\star$ Take now $\eta > 0$ in (4.A.8). We see that $\hat{Z}$ jumps only at $t = 0$ with size η, so that

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge h} e^{-\rho t}(\rho\varphi-\mathcal{L}_i\varphi)(X_t^{\bar{x},i})dt\right]-\eta-(\varphi(\overline{x}-\eta)-\varphi(\overline{x}))\ge 0. \qquad (4.A.10)$$

By sending $h \to 0$, and then dividing by η and letting $\eta \to 0$, we obtain

$$\varphi'(\overline{x})-1\ge 0. \qquad (4.A.11)$$

This proves the required supersolution property

$$\min\left[(\rho\varphi-\mathcal{L}_i\varphi)(\overline{x}),\ \varphi'(\overline{x})-1, v_i(\overline{x})-v_{1-i}(\overline{x}-g_{i,1-i})\right]\ge 0. \qquad (4.A.12)$$

Proof of the subsolution property.
We prove the subsolution property by contradiction. Suppose that the claim is not true. Then, there exists $\overline{x} > 0$ and a neighborhood $B_\varepsilon(\overline{x}) = (\overline{x}-\varepsilon, \overline{x}+\varepsilon)$ of $\overline{x}$, $\overline{x} > \varepsilon > 0$, a C^2 function φ with $(\varphi - v_*)(\overline{x}) = 0$ and $\varphi \ge v_i$ on $\overline{B}_\varepsilon(\overline{x})$, and $\eta > 0$, s.t. for all $x \in \overline{B}_\varepsilon(\overline{x})$:

$$\begin{aligned}\rho\varphi(x)-\mathcal{L}_i\varphi(x) &> \eta, &(4.A.13)\\ \varphi'(x)-1 &> \eta, &(4.A.14)\\ v_i(x)-v_{i-1}(x-g_{i,1-i}) &> \eta. &(4.A.15)\end{aligned}$$

For any admissible control $\alpha = (Z, \tau_n, n \ge 1)$, consider the exit time $\tau_\varepsilon = \inf\{t \ge 0, X_t^{\bar{x},i} \notin \overline{B}_\varepsilon(\overline{x})\}$. We notice that $\tau_\varepsilon < T$. Applying Itô's formula to the process $e^{-\rho t}\varphi(X_t^{\bar{x},i})$ between 0 and $(\tau_\varepsilon \wedge \tau_1)^-$, and by noting that before $(\tau_\varepsilon \wedge \tau_1)^-$, $X^{x,i}$ stays in regime i and in the ball $\overline{B}_\varepsilon(\overline{x})$, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-\rho(\tau_\varepsilon\wedge\tau_1)^-}\varphi(X_{(\tau_\varepsilon\wedge\tau_1)^-}^{\bar{x},i})\right] &= \varphi(\overline{x})+\mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}(-\rho\varphi(X_t^{\bar{x},i})+\mathcal{L}_i\varphi(X_t^{\bar{x},i}))dt\right]\\ &\quad-\mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}\varphi'(X_t^{\bar{x},i})dZ_t^c\right]\\ &\quad+\mathbb{E}\left[\sum_{0\le t<\tau_\varepsilon\wedge\tau_1} e^{-\rho t}[\varphi(X_t^{\bar{x},i})-\varphi(X_{t^-}^{\bar{x},i})]\right]. \qquad (4.A.16)\end{aligned}$$

From Taylor's formula and (4.A.14), and noting that $\Delta X_t^{\bar{x},i} = -\Delta Z_t$ for all $0 \leq t < \tau_\varepsilon \wedge \tau_1$, we have

$$\begin{aligned} \varphi(X_t^{\bar{x},i}) - \varphi(X_{t^-}^{\bar{x},i}) &= \Delta X_t^{\bar{x},i}\,\varphi'(X_t^{\bar{x},i} + z\Delta X_t^{\bar{x},i}) \\ &\leq -(1+\eta)\Delta Z_t \end{aligned} \tag{4.A.17}$$

Plugging the relations (4.A.13), (4.A.14), and (4.A.17) into (4.A.16), we obtain

$$\begin{aligned} v_i(\overline{x}) = \varphi(\overline{x}) \;&\geq\; \mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t} dZ_t + e^{-\rho(\tau_\varepsilon\wedge\tau_1)^-}\varphi(X^{\bar{x},i}_{(\tau_\varepsilon\wedge\tau_1)^-})\right] \\ &\quad +\eta\left(\mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dt\right] + \mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dZ_t\right]\right). \\ &\geq\; \mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t} dZ_t + e^{-\rho\tau_\varepsilon^-}\varphi(X^{\bar{x},i}_{\tau_\varepsilon^-})1_{\tau_\varepsilon<\tau_1} + e^{-\rho\tau_1^-}\varphi(X^{\bar{x},i}_{\tau_1^-})1_{\tau_1\leq\tau_\varepsilon}\right] \\ &\quad +\eta\left(\mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dt\right] + \mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dZ_t\right]\right). \end{aligned} \tag{4.A.18}$$

Notice that while $X^{\bar{x},i}_{\tau_\varepsilon^-} \in \overline{B}_\varepsilon(\overline{x})$, $X^{\bar{x},i}_{\tau_\varepsilon}$ is either on the boundary $\partial\overline{B}_\varepsilon(\overline{x})$ or out of $\overline{B}_\varepsilon(\overline{x})$. However, there is some random variable γ valued in $[0,1]$ s.t.

$$\begin{aligned} X^{(\gamma)} &= X^{\bar{x},i}_{\tau_\varepsilon^-} + \gamma\Delta X^{\bar{x},i}_{\tau_\varepsilon} \\ &= X^{\bar{x},i}_{\tau_\varepsilon^-} - \gamma\Delta Z_{\tau_\varepsilon} \;\in\; \partial\overline{B}_\varepsilon(\overline{x}). \end{aligned}$$

Then similarly as in (4.A.17), we have

$$\varphi(X^{(\gamma)}) - \varphi(X^{\bar{x},i}_{\tau_\varepsilon^-}) \quad\leq\quad -\gamma(1+\eta)\Delta Z_{\tau_\varepsilon}. \tag{4.A.19}$$

Noting that $X^{(\gamma)} = X^{\bar{x},i}_{\tau_\varepsilon} + (1-\gamma)\Delta Z_{\tau_\varepsilon}$, we have

$$v_i(X^{(\gamma)}) \quad\geq\quad v_i(X^{\bar{x},i}_{\tau_\varepsilon}) + (1-\gamma)\Delta Z_{\tau_\varepsilon}. \tag{4.A.20}$$

Recalling that $\varphi(X^{(\gamma)}) \geq v_i(X^{(\gamma)})$, inequalities (4.A.19) and (4.A.20) imply

$$\varphi(X_{\tau_\varepsilon^-}) \geq v_i(X^{\bar{x},i}_{\tau_\varepsilon}) + (1+\gamma\eta)\Delta Z_{\tau_\varepsilon}$$

Plugging into (4.A.18) and using (4.A.15), we have

$$\begin{aligned} v_i(\overline{x}) \;&\geq\; \mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dZ_t + e^{-\rho\tau_\varepsilon}v_i(X^{\bar{x},i}_{\tau_\varepsilon})1_{\tau_\varepsilon<\tau_1} + e^{-\rho\tau_1}v_{1-i}(X^{\bar{x},i}_{\tau_1})1_{\tau_1\leq\tau_\varepsilon}\right] \\ &\quad + \eta\,\mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge\tau_1} e^{-\rho t}dt + \int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dZ_t + e^{-\rho\tau_1}1_{\tau_1\leq\tau_\varepsilon} + \gamma e^{-\rho\tau_\varepsilon\wedge\tau_1}\Delta Z_{\tau_\varepsilon}1_{\tau_\varepsilon<\tau_1}\right] \\ &\quad + \mathbb{E}\left[e^{-\rho\tau_\varepsilon}\Delta Z_{\tau_\varepsilon}1_{\tau_\varepsilon<\tau_1}\right]. \end{aligned} \tag{4.A.21}$$

We now claim that there exists a constant $c_0 > 0$ such that for any admissible control

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge\tau_1} e^{-\rho t}dt + \int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dZ_t + e^{-\rho\tau_1}1_{\tau_1\le\tau_\varepsilon} + \gamma e^{-\rho\tau_\varepsilon\wedge\tau_1}\Delta Z_{\tau_\varepsilon}1_{\tau_\varepsilon<\tau_1}\right] \ge c_0. \quad (4.A.22)$$

The C^2 function $\psi(x) = c_0\left[1 - \frac{(x-\overline{x})^2}{\varepsilon^2}\right]$, with

$$0 < c_0 \le \min\left\{\left(\rho + \frac{2}{\varepsilon}\mu_i + \frac{1}{\varepsilon^2}\sigma^2\right)^{-1}, \frac{\varepsilon}{2}\right\}.$$

satisfies

$$\begin{cases} \min\{-\rho\psi + \mathcal{L}_i\psi + 1, 1 - \psi', -\psi + 1\} & \ge & 0, & \text{on } \overline{B}_\varepsilon(\overline{x}), \\ \psi & = & 0, & \text{on } \partial\overline{B}_\varepsilon(\overline{x}),. \end{cases} \quad (4.A.23)$$

Applying Itô's formula, we then obtain

$$\begin{aligned} 0 < c_0 = \psi(\overline{x}) \quad &\le \quad \mathbb{E}\left[e^{-\rho(\tau_\varepsilon\wedge\tau_1)}\psi(X^{\bar{x},i}_{(\tau_\varepsilon\wedge\tau_1)^-})\right] \\ &\quad + \mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge\tau_1} e^{-\rho t}dt\right] + \mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dZ_t\right]. \end{aligned} \quad (4.A.24)$$

Noting that $\psi'(x) \le 1$, we have

$$\psi(X^{\bar{x},i}_{\tau_\varepsilon^-}) - \psi(X^{(\gamma)}) \quad \le \quad (X^{\bar{x},i}_{\tau_\varepsilon^-} - X^{(\gamma)}) = \gamma\Delta Z_{\tau_\varepsilon}$$

Plugging into (4.A.24), we obtain

$$\begin{aligned} 0 < c_0 \quad &\le \quad \mathbb{E}\left[e^{-\rho\tau_1}\psi(X^{\bar{x},i}_{\tau_1^-})1_{\tau_1\le\tau_\varepsilon}\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon\wedge\tau_1} e^{-\rho t}dt\right] \\ &\quad + \mathbb{E}\left[\int_0^{(\tau_\varepsilon\wedge\tau_1)^-} e^{-\rho t}dZ_t\right] + \mathbb{E}\left[\gamma e^{-\rho\tau_\varepsilon}\Delta Z_{\tau_\varepsilon}1_{\tau_\varepsilon<\tau_1}\right]. \end{aligned} \quad (4.A.25)$$

Since $\psi(x) \le 1$ for all $x \in B_\varepsilon(\overline{x})$, this proves the claim (4.A.22).

Finally, by taking the supremum over all admissible control α, and using the dynamic programming principle **(DP)**, (4.A.21) implies $v_i(\overline{x}) \ge v_i(\overline{x}) + \eta c_0$, which is a contradiction. Thus we obtain the required viscosity subsolution property :

$$\min\left[(\rho\varphi - \mathcal{L}_i\varphi)(\overline{x}),\ \varphi'(\overline{x}) - 1, v_i(\overline{x}) - v_{i-1}(\overline{x} - g_{i,i-1})\right] \le 0. \quad (4.A.26)$$

□

Proof of the uniqueness property.

Suppose u_i, $i = 0, 1$, are continuous viscosity subsolutions to the system of variational inequalities on $(0, \infty)$, and w_i, $i = 0, 1$, continuous viscosity supersolutions to the system of variational inequalities on $(0, \infty)$, satisfying the boundary conditions $u_i(0^+) \le w_i(0^+)$, $i = 0, 1$, and the linear growth condition :

$$|u_i(x)| + |w_i(x)| \le C_1 + C_2 x, \quad \forall x \in (0, \infty),\ i = 1, 2, \quad (4.A.27)$$

for some positive constants C_1 and C_2. We want to prove that

$$u_i \leq w_i, \quad \text{on} \quad (0,\infty),\ i = 0,1,$$

Step 1. We first construct strict supersolutions to the system with suitable perturbations of w_i, $i = 0,1$. We set

$$h_i(x) = A_i + B_i x + Cx^2, \quad x > 0,$$

where

$$\begin{aligned}
A_0 &= \frac{\mu_1 B_1 + C\sigma^2 + 1}{\rho} + \frac{C}{4}\left(\frac{B_1}{C} - 2\frac{\mu_1}{\rho}\right)^2 + \frac{C}{4}\left(\frac{B_0}{C} - 2\frac{\mu_0}{\rho}\right)^2 + w_0(0^+) + w_1(0^+),\\
A_1 &= A_0 + \frac{3}{2}g + \frac{g}{\lambda},\\
B_0 &= 3,\ B_1 = 2 + \frac{2}{\lambda},\\
C &= \frac{1}{\lambda g}.
\end{aligned}$$

We then define for all $\gamma \in (0,1)$, the continuous functions on $(0,\infty)$ by :

$$w_i^\gamma = (1-\gamma)w_i + \gamma h_i, \quad i = 0,1.$$

We then see that for all $\gamma \in (0,1), i = 0,1$:

$$\begin{aligned}
w_i^\gamma(x) - w_{1-i}^\gamma(x - g_{i,1-i}) &= (1-\gamma)\left[w_i(x) - w_{1-i}(x - g_{i,1-i})\right] + \gamma\left[h_i(x) - h_{1-i}(x - g_{i,1-i})\right],\\
&\geq \gamma\left[(2Cg_{i,1-i} + B_i - B_{1-i})x + A_i - A_{1-i} - Cg_{i,1-i}^2 + B_{1-i}g_{i,1-i}\right],\\
&\geq \gamma\frac{g}{2}, \quad i = 0,1. \qquad (4.A.28)
\end{aligned}$$

Furthermore, we also easily obtain

$$h_i'(x) - 1 \quad = \quad B_i + 2Cx - 1 \geq 1. \qquad (4.A.29)$$

A straight calculation will also provide us with the last required inequality, i.e.

$$\rho h_i(x) - \mathcal{L}_i h_i(x) \geq 1. \qquad (4.A.30)$$

Combining (4.A.28), (4.A.29), and (4.A.30), this shows that w_i^γ is a strict supersolution of the system : for $i = 0,1$, we have on $(0,\infty)$

$$\min\left[\rho w_i^\gamma(x) - \mathcal{L}_i w_i^\gamma(x),\ w_i^{\gamma\prime}(x) - 1,\ w_i^\gamma(x) - w_{i-1}^\gamma(x - g_{i,1-i})\right] \geq \gamma\min\{1, \frac{g}{2}\} = \delta \qquad (4.A.31)$$

Step 2. In order to prove the comparison principle, it suffices to show that for all $\gamma \in (0,1)$:

$$\max_{i\in\{0,1\}} \sup_{(0,+\infty)} (u_i - w_i^\gamma) \leq 0,$$

since the required result is obtained by letting γ to 0. We argue by contradiction and suppose that there exist some $\gamma \in (0,1)$ and $i \in \{0,1\}$, s.t.

$$\theta := \max_{j\in\{0,1\}} \sup_{(0,+\infty)} (u_j - w_j^\gamma) = \sup_{(0,+\infty)} (u_i - w_i^\gamma) > 0. \tag{4.A.32}$$

Notice that $u_i(x) - w_i^\gamma(x)$ goes to $-\infty$ when x goes to infinity. We also have $\lim_{x\to 0^+} u_i(x) - \lim_{x\to 0^+} w_i^\gamma(x) \le \gamma(\lim_{x\to 0^+} w_i(x) - A_i) \le 0$. Hence, by continuity of the functions u_i and w_i^γ, there exists $x_0 \in (0,\infty)$ s.t.

$$\theta = u_i(x_0) - w_i^\gamma(x_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned} \Phi_\varepsilon(x,y) &= u_i(x) - w_i^\gamma(y) - \phi_\varepsilon(x,y), \\ \phi_\varepsilon(x,y) &= \frac{1}{4}|x-x_0|^4 + \frac{1}{2\varepsilon}|x-y|^2, \end{aligned}$$

for all $x, y \in (0,\infty)$. By standard arguments in comparison principle, the function Φ_ε attains a maximum in $(x_\varepsilon, y_\varepsilon) \in (0,\infty)^2$, which converges (up to a subsequence) to (x_0, x_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{4.A.33}$$

Applying Theorem 3.2 in [19], we get the existence of M_ε, $N_\varepsilon \in \mathbb{R}$ such that:

$$\begin{aligned} (p_\varepsilon, M_\varepsilon) &\in J^{2,+}u_i(x_\varepsilon), \\ (q_\varepsilon, N_\varepsilon) &\in J^{2,-}w_i^\gamma(y_\varepsilon), \end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & N_\varepsilon \end{pmatrix} \le D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + \varepsilon(D^2\phi(x_\varepsilon, y_\varepsilon))^2, \tag{4.A.34}$$

where

$$\begin{aligned} p_\varepsilon &= D_x\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3, \\ q_\varepsilon &= -D_y\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), \\ D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) &= \begin{pmatrix} 3(x_\varepsilon - x_0)^2 + \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix}. \end{aligned}$$

By writing the viscosity subsolution property of u_i and the viscosity supersolution property (4.A.31) of w_i^γ, we have the following inequalities:

$$\min\Bigg\{\rho u_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right)\mu_i - \frac{1}{2}\sigma^2 M_\varepsilon,$$
$$\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) - 1,\ u_i((x_\varepsilon) - u_{1-i}(x_\varepsilon - g_{i,1-i})\Bigg\} \le 0, \tag{4.A.35}$$

$$\min\Bigg\{\rho w_i^\gamma(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)\mu_i - \frac{1}{2}\sigma^2 N_\varepsilon,\ \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 1,$$
$$w_i^\gamma(y_\varepsilon) - w_{i-1}^\gamma(x_\varepsilon - g_{i,1-i})\Bigg\} \ge \delta. \tag{4.A.36}$$

We then distinguish the following three cases :

⋆ *Case 1* : $u_i(x_\varepsilon) - u_{1-i}(x_\varepsilon - g_{i,1-i}) \leq 0$ in (4.A.35).
From the continuity of u_i and by sending $\varepsilon \to 0$, this implies

$$u_i(x_0) \leq u_{1-i}(x_0 - g_{i,1-i}). \tag{4.A.37}$$

On the other hand, from (4.A.36), we also have

$$w_i^\gamma(y_\varepsilon) - w_{i-1}^\gamma(x_\varepsilon - g_{i,1-i}) \geq \delta,$$

which implies, by sending $\varepsilon \to 0$ and using the continuity of w_i :

$$w_i^\gamma(x_0) \geq w_{i-1}^\gamma(x_0 - g_{i,1-i}) + \delta. \tag{4.A.38}$$

Combining (4.A.37) and (4.A.38), we obtain

$$\begin{aligned} \theta = u_i(x_0) - w_i^\gamma(x_0) &\leq u_{1-i}(x_0 - g_{i,1-i}) - w_{i-1}^\gamma(x_0 - g_{i,1-i}) - \delta, \\ &\leq \theta - \delta, \end{aligned}$$

which is a contradiction.

⋆ *Case 2* : $\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) - 1 \leq 0$ in (4.A.35)
Notice that by (4.A.36), we have

$$\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 1 \geq \delta,$$

which implies in this case

$$(x_\varepsilon - x_0)^3 \leq -\delta.$$

By sending ε to zero, we obtain again a contradiction.

⋆ *Case 3* : $\rho u_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right)\mu_i - \frac{1}{2}\sigma^2 M_\varepsilon \leq 0$ in (4.A.35)
From (4.A.36), we have

$$\rho w_i^\gamma(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)\mu_i - \frac{1}{2}\sigma^2 N_\varepsilon \geq \delta,$$

which implies in this case

$$\rho\left(u_i(x_\varepsilon) - w_i^\gamma(y_\varepsilon)\right) - \mu_i(x_\varepsilon - x_0)^3 - \frac{1}{2}\sigma^2(M_\varepsilon - N_\varepsilon) \leq -\delta, \tag{4.A.39}$$

From (4.A.34), we have

$$\frac{1}{2}\sigma^2(M_\varepsilon - N_\varepsilon) \leq \frac{3}{2}\sigma^2(x_\varepsilon - x_0)^2\left[1 + 3\varepsilon(x_\varepsilon - x_0)\right].$$

Plugging it into (4.A.39) yields

$$\rho\left(u_i(x_\varepsilon) - w_i^\gamma(y_\varepsilon)\right) \leq \mu_i(x_\varepsilon - x_0)^3 + \frac{3}{2}\sigma^2(x_\varepsilon - x_0)^2\left[1 + 3\varepsilon(x_\varepsilon - x_0)\right] - \delta.$$

By sending ε to zero and using the continuity of u_i and w_i^γ, we obtain the required contradiction : $\rho\theta \leq -\delta < 0$. This ends the proof. □

Appendix B : Proof of Proposition 4.3.3

C^1 property

We prove in three steps that for a given $i \in 0, 1$, v_i is a C^1 function on $(0, \infty)$. Notice first that since v_i is a strictly nondecreasing continuous function on $(0, \infty)$, it admits a nonnegative left and right derivative $v_i'^-(x)$ and $v_i'^+(x)$ for all $x > 0$.

Step 1. We start by proving that $v_i'^-(x) \geq v_i'^+(x)$ for all $x \in (0, \infty)$.
Suppose on the contrary that there exists some x_0 such that $v_i'^-(x_0) < v_i'^+(x_0)$. Take then some $q \in (v_i'^-(x), v_i'^+(x))$, and consider the function

$$\varphi(x) = v_i(x_0) + q(x - x_0) + \frac{1}{2\varepsilon}(x - x_0)^2,$$

with $\varepsilon > 0$. Then x_0 is a local minimum of $v_i - \varphi_i$, with $\varphi'(x_0) = q$ and $\varphi''(x_0) = \frac{1}{\varepsilon}$. Therefore, we get the required contradiction by writing the supersolution inequality :

$$0 \leq \rho v_i(x_0) - \mu_i \varphi'(x_0) - \frac{\sigma^2}{2}\varphi''(x_0) = \rho v_i(x_0) - \mu_i q - \frac{\sigma^2}{2\varepsilon},$$

and choosing ε small enough.

Step 2. We now prove that for $i = 0, 1$, v_i is C^1 on $(0, \infty) \backslash \mathcal{S}_i$.
Suppose there exists some $x_0 \notin \mathcal{S}_i$ s.t. $v_i'^-(x_0) > v_i'^+(x_0)$. We then fix some $q \in (v_i'^+(x_0), v_i'^-(x_0))$ and consider the function

$$\varphi(x) = v_i(x_0) + q(x - x_0) - \frac{1}{2\varepsilon}(x - x_0)^2,$$

with $\varepsilon > 0$. Then x_0 is a local maximum of $v_i - \varphi$, with $\varphi'(x_0) = q > 1$, $\varphi''(x_0) = -\frac{1}{\varepsilon}$. Since $x_0 \notin \mathcal{S}_i$, the subsolution inequality property implies :

$$\rho v_i(x_0) - \mu_i q + \frac{\sigma^2}{2\varepsilon} \leq 0,$$

which leads to a contradiction, by choosing ε sufficiently small. By combining the results from step 1 and step 2, we obtain that v_i is C^1 on the open set $(0, \infty) \backslash \mathcal{S}_i$.

Step 3. We now prove that v_i is C^1 on $(0, \infty)$.
From Step 2, we have to prove the C^1 property of v_i on $\mathcal{S}_i^*$. Fix then some $x_0 \in \mathcal{S}_i^*$ so that $v_i(x_0) = v_{1-i}(x_0 - g_{i,1-i})$. Hence, x_0 is a minimum of $v_i - v_{1-i}(. - g_{i,1-i})$, and so

$$v_i'^-(x_0) - v_{1-i}'^-(x_0 - g_{i,1-i}) \leq v_i'^+(x_0) - v_{1-i}'^+(x_0 - g_{i,1-i}). \tag{4.B.1}$$

Now, from Lemma 4.4.3, $x_0 - g_{i,1-i}$ belongs to the open set $(0, \infty) \backslash \mathcal{S}_{1-i}$. From step 2, v_{1-i} is C^1 on $(0, \infty) \backslash \mathcal{S}_{1-i}$, and so $v_{1-i}'^+(x_0 - g_{i,1-i}) = v_{1-i}'^-(x_0 - g_{i,1-i})$. From (4.B.1), we thus obtain

$$v_i'^-(x_0) \leq v_i'^+(x_0),$$

which is the required result, since the reverse inequality is already satisfied from Step 1.

C^2 property

We now turn to the proof of the C^2 property of v_i on the open set $\mathcal{C}_i \cup \mathcal{D}_i$ of $(0, \infty)$.
Step 1. First, we prove that v_i is C^2 on $\mathcal{C}_i$. By standard arguments, we check that v_i is a viscosity solution to

$$\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad x \in \mathcal{C}_i. \tag{4.B.2}$$

Indeed, let $\overline{x} \in \mathcal{C}_i$ and φ a C^2 function on $\mathcal{C}_i$ s.t. $\overline{x}$ is a local maximum of $v_i - \varphi$, with $v_i(\overline{x}) = \varphi(\overline{x})$. Then, $\varphi'(\overline{x}) = v_i'(\overline{x}) > 1$. By definition of $\mathcal{C}_i$, we also have $v_i(\overline{x}) > v_{1-i}(x - g_{i,1-i})$ and so from the subsolution viscosity property (4.A.26) of v_i, we have

$$\rho\varphi(\overline{x}) - \mathcal{L}_i\varphi(\overline{x}) \leq 0.$$

The supersolution inequality for (4.B.2) is immediate from (4.A.12).
Now, for any arbitrary bounded interval $(x_1, x_2) \subset \mathcal{C}_i$, consider the Dirichlet boundary linear problem:

$$\rho w(x) - \mathcal{L}_i w(x) = 0, \quad \text{on } (x_1, x_2) \tag{4.B.3}$$

$$w(x_1) = v_i(x_1), \quad w(x_2) = v_i(x_2). \tag{4.B.4}$$

Classical results provide the existence and uniqueness of a smooth C^2 function w solution on (x_1, x_2) to (4.B.3)-(4.B.4). In particular, this smooth function w is a viscosity solution to (4.B.2) on (x_1, x_2). From standard uniqueness results for (4.B.3)-(4.B.4), we get $v_i = w$ on (x_1, x_2). From the arbitrariness of $(x_1, x_2) \subset \mathcal{C}_i$, this proves that v_i is smooth C^2 on $\mathcal{C}_i$.

Step 2. It is clear that v_i is C^2 on $\mathcal{D}_i$. We now prove that v_i is C^2 on $\mathcal{C}_i \cup \mathcal{D}_i$, i.e. v_i is also C^2 on any point $x_0 \in \overline{\mathcal{C}_i} \cap \overline{\mathcal{D}_i}$. We need to prove that $\lim_{x \uparrow x_0^-} v''{}_i(x) = \lim_{x \downarrow x_0^+} v''{}_i(x) = 0$. We set

$$x_a = \inf\{x \leq x_0, (x, x_0) \subset \mathcal{C}_i \cup \mathcal{D}_i\}, \qquad x_b = \sup\{x \geq x_0, (x_0, x) \subset \mathcal{C}_i \cup \mathcal{D}_i\},$$

and we distinguish the three following cases :
$\star$ *Case 1* : $(x_a, x_0) \subset \mathcal{C}_i$ and $[x_0, x_b) \subset \mathcal{D}_i$.
By definition of $\mathcal{D}_i$, we then have for all $x \in [x_0, x_b)$, $v_i(x) = x - x_0 + v_i(x_0)$. From the viscosity supersolution property of v_i, this implies

$$\rho(x - x_0) + \rho v_i(x_0) - \mu_i \geq 0, \quad \forall x \in (x_0, x_b).$$

Hence, by sending $x \downarrow x_0^+$, we obtain

$$\rho v_i(x_0) - \mu_i \geq 0. \tag{4.B.5}$$

On the other hand, from the above step 1, we also know that v_i is a classical C^2 solution to the equation (4.B.2), and so

$$\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad \forall x \in (x_a, x_0). \tag{4.B.6}$$

By sending $x \uparrow x_0^-$, and using the fact that v_i is C^1 on $(0, \infty)$, in particular, $v_i'(x_0^-) = v_i'(x_0^+) = 1$, we obtain the existence of $v_i''(x_0^-)$ s.t.

$$\rho v_i(x_0) - \mu_i - \frac{\sigma^2}{2} v''{}_i(x_0^-) = 0. \tag{4.B.7}$$

Plugging it into (4.B.5), we obtain $v_i''(x_0^-) \geq 0 = v_i''(x_0^+)$. Suppose now that $v_i''(x_0^-) > 0$. This leads to a contradiction since it would mean that v_i' is strictly non decreasing in a left neighbourhood of x_0, i.e. $v_i' < v_i'(x_0) = 1$, which is impossible given the viscosity supersolution property. Therefore, $v''(x_0)$ exists and v'' is continuous on x_0.

$\star$ *Case 2* : $(x_a, x_0] \subset \mathcal{D}_i \cap \mathcal{S}_i$ and $(x_0, x_b) \subset \mathcal{C}_i$.
We show actually that this case is impossible. Indeed, we would have

$$v_i(x) = v_{1-i}(x - g_{i,1-i}) = x - x_0 + v_i(x_0), \quad \forall x \in (x_a, x_0],$$

and so $(x_a - g_{i,1-i}, x_0 - g_{i,1-i}] \subset \mathcal{D}_{1-i}$. Hence, from the viscosity supersolution property of v_i and v_{1-i}, this would imply

$$\rho v_{1-i}(x_0 - g_{i,1-i}) - \mu_{1-i} \geq 0, \tag{4.B.8}$$

$$\rho v_i(x_0) - \mu_i \geq 0. \tag{4.B.9}$$

We then consider the functions w_i and w_{1-i} :

$$w_i(x) = \begin{cases} x - x_0 + v_i(x_0), & x > x_0 \\ v_i(x), & x \leq x_0, \end{cases}$$

$$w_{1-i}(x) = \begin{cases} x - (x_0 - g_{i,1-i}) + v_{1-i}(x_0 - g_{i,1-i}), & x > x_0 - g_{i,1-i}, \\ v_{1-i}(x), & x \leq x_0 - g_{i,1-i}. \end{cases}$$

We now claim that $(w_j)_{j \in \{i,1-i\}}$ are viscosity solutions on $(0, \infty)$ to

$$\min\left[\rho w_j(x) - \mathcal{L}_j w_j(x), w_j'(x) - 1, w_j(x) - w_{1-j}(x - g_{j,1-j})\right] = 0, \quad x > 0, \tag{4.B.10}$$

Let us check the viscosity solution property of w_i. Take some $\overline{x} \in (0, \infty)$ and φ a C^2 function s.t $\overline{x}$ is local minimum of $w_i - \varphi$ with $w_i(\overline{x}) = \varphi(\overline{x})$. If $\overline{x} \leq x_0$, given the definition of w_i and noting that $w_i \leq v_i$, we obtain that $\overline{x}$ is also a local minimum of $v_i - \varphi$ with $v_i(\overline{x}) = \varphi(\overline{x})$. From the viscosity supersolution of v_i and observing that $v_{1-i} \geq w_{1-i}$, this yields

$$\min\left[\rho\varphi(\overline{x}) - \mathcal{L}_i\varphi(\overline{x}), \varphi'(\overline{x}) - 1, \varphi(\overline{x}) - w_{1-i}(\overline{x} - g_{i,1-i})\right] \geq 0.$$

If $\overline{x} > x_0$, we have

$$\begin{aligned} \varphi'(\overline{x}) &= w_i'(\overline{x}) = 1, && (4.B.11)\\ \varphi''(\overline{x}) &\leq w_i''(\overline{x}) = 0, && (4.B.12)\\ w_i(\overline{x}) &= w_{1-i}(\overline{x}). && (4.B.13) \end{aligned}$$

From (4.B.11)-(4.B.12), we have after straightforward calculation :

$$\rho\varphi(\overline{x}) - \mathcal{L}_i\varphi(\overline{x}) \geq \rho\varphi(\overline{x}) - \mu_i \ = \ \rho v_i(\overline{x}) - \mu_i \ \geq \ 0, \quad (4.B.14)$$

by (4.B.9). Hence, from (4.B.11), (4.B.13), and (4.B.14), we obtain

$$\min\left[\rho\varphi(\overline{x}) - \mathcal{L}_i\varphi(\overline{x}), \varphi'(\overline{x}) - 1, \varphi(\overline{x}) - w_{1-i}(\overline{x} - g_{i,1-i})\right] \geq 0,$$

which proves the viscosity supersolution property of w_i to (4.B.10).
We now turn to the viscosity subsolution property of w_i. Let $\overline{x} \in (0, \infty)$ and φ a C^2 s.t $\overline{x}$ is a local maximum of $w_i - \varphi$. If $\overline{x} < x_0$ and using the definition of w_i, we obtain the required desired subsolution property. If $\overline{x} \geq x_0$, we have $\varphi'(\overline{x}) = w_i'(\overline{x}) = 1$, and so the desired subsolution property.

By proceeding in the same manner, we also obtain that w_{1-i} is a viscosity solution to (4.B.10). Hence, w_j, $j = 0, 1$, satisfy the same boundary constraints and linear growth as v_j, $j = 0, 1$, which proves, by uniqueness property, that $w_j = v_j$, $j = 0, 1$. This is a contradiction given the definition of w_j.

$\star$ *Case 3* : $(x_a, x_0] \subset \mathcal{D}_i$ with $(x_a, x_0] \cap \mathcal{S}_i = \emptyset$, and $(x_0, x_b) \subset \mathcal{C}_i$.
In this case, we distinguish two separate possibilities:
- **(i)** $x_0 + g_{1-i,i} \in \mathcal{S}_{1-i}$
Let $x \in (x_a, x_0]$, we have

$$\begin{aligned} v_i(x) &= x - x_0 + v_i(x_0),\\ v_{1-i}(x_0 + g_{1-i,i}) &\geq x_0 - x + v_{1-i}(x + g_{1-i,i}), && (4.B.15)\\ &\geq x_0 - x + v_i(x) && (4.B.16) \end{aligned}$$

But we also have $v_i(x_0) = v_{1-i}(x_0 + g_{1-i,i})$. As such, the inequalities in (4.B.15)-(4.B.16) become equalities, and

$$\begin{aligned} v_{1+i}(x + g_{1-i,i}) = x - x_0 + v_{1-i}(x_0 + g_{1-i,i})\\ \text{i.e.} \quad (x_a + g_{1-i,i}, x_0 + g_{1-i,i}] \subset \mathcal{D}_{1-i} \end{aligned}$$

We now consider the following functions:

$$w_i(x) \ = \ \begin{cases} x - x_0 + v_i(x_0), & x > x_0\\ v_i(x), & x \leq x_0, \end{cases}$$

$$w_{1-i}(x) \quad = \quad \begin{cases} x-(x_0+g_{1-i,i})+v_{1-i}(x_0+g_{1-i,i}), & x > x_0+g_{1-i,i}, \\ v_{1-i}(x), & x \leq x_0+g_{1-i,i}. \end{cases}$$

By the same arguments as above, we obtain that $(w_j)_{j=0,1}$ are viscosity solutions to

$$\min\left[\rho u_j(x) - \mathcal{L}_j u_j(x), u_j'(x)-1, u_j(x) - u_{1-j}(x-g_{j,1-j})\right] = 0, \quad \forall x > 0,$$

and due to the uniqueness property, $w_i = v_i$ for $i = 0, 1$, which is a contradiction given the definition of w_i.

- **(ii)** $x_0 + g_{1-i,i} \notin \mathcal{S}_{1-i}$

Let $x_1 = \inf\{x \geq x_0 + g_{1-i,i}, x \notin \mathcal{S}_{1-i}\}$. By the continuity of $(v_i)_{i=0,1}$, we have $x_1 > x_0 + g_{1-i,i}$. Recall that $v_i(x) = x - x_0 + v_i(x_0)$ for all $x \in (x_a, x_0] \subset \mathcal{D}_i$. Moreover, since $x_0 \notin \mathcal{S}_i$, we have $v_i(x_0) > v_{1-i}(x_0 - g_{i,1-i})$. Consider the function $h(x) = x - x_0 + v_i(x_0) - v_{1-i}(x - g_{i,1-i})$ so that $h(x_0) > 0$. Let $x_2 = \sup\{x \geq x_0, h(x) \geq 0\}$. Given the continuity of h, we have $x_2 > x_0$. We fix $x_c = \min\{x_1 - g_{1-i,i}, x_2\}$. We now consider the following functions:

$$w_i(x) \quad = \quad \begin{cases} x - x_0 + v_i(x_0), & x \in (x_0, x_c), \\ v_i(x), & x \notin (x_0, x_c), \end{cases}$$

$$w_{1-i} \quad = \quad v_{1-i}, \quad \text{on } (0, \infty)$$

We may easily prove that $(w_j)_{j=0,1}$ are viscosity solutions to (4.3.21), which implies from uniqueness property that $w_j = v_j$, $j = 0, 1$, a contradiction.

We therefore conclude that v_i is C^2 on $\mathcal{C}_i \cup \mathcal{D}_i$, which ends the proof. $\square$

Part III

ASYMMETRIC INFORMATION

Chapter 5

Competitive market equilibrium under asymmetric information

In revision with *Decisions in Economics and Finance.*

Abstract : This paper studies the existence of competitive market equilibrium under asymmetric information. There are two agents involved in the trading of the risky asset: an "informed" trader and an "ordinary" trader. The market is competitive and the ordinary agent can infer the insider information from the risky asset's price dynamics. The definition of market equilibrium is based on the law of supply-demand as described by a Rational Expectations Equilibrium of the Grossman and Stiglitz (1980) model. We show that equilibrium can be attained by linear dynamics of an admissible price process of the risky asset for a given linear supply dynamics.

Keywords: insider trading, stochastic filtering theory, equilibrium, utility maximization.

5.1 Introduction

In recent years, financial mathematicians have been focusing on the model of asymmetric information. Asymmetric information arises when agents in the market do not have the same information filtration. They generally make an assumption regarding the extra information that is accessible uniquely by the "informed trader" or the "insider trader". This extra information could be, for example, the future liquidation price of the risky asset. Using the results of enlargement of filtration first developed by Jeulin [44] and then Jacod [42], many papers such as those of Pikovsky and Karatzas and Grorud and Pontier [33] focused on solving utility maximization problems in a security market where two investors have different information levels. In these papers, the security prices are assumed to evolve according to an exogenous diffusion. In Hillairet [39], different types of asymmetric information, including "initial strong", "progressive strong" and "weak" information are studied. However, the drawback of the above models is that "ordinary" or "uninformed" agents cannot infer the insider information.

On the other hand, in Kyle [48] and Back [3], the market is competitive and the ordinary agents can obtain feedbacks from the market regarding the insider information. There have also been several other studies, published in the economic literature, on the impact of asymmetric information on stock price. The first such paper is the seminal paper of Grossman-Stiglitz [34], followed by those of Glosten-Milgrom [32]. In Biais-Rochet [8], we may find a very insightful survey of the literature on these areas, including those cited above. In [34], the agents are competitive and market is Walrasian, i.e. price equals supply and demand. The only exogenous part of this model may come from irrational traders, often called noise traders. In [8], the objective is to analyse the price formation in a dynamic version of Grossman and Stiglitz model where stochastic control techniques can be used.

In the same framework, in our paper, we consider a financial market consisting of two traders, an "ordinary" agent and an "informed" agent and noise traders. While the ordinary agent can only observe the risky asset's price dynamics, the "informed" agent has also access to the total supply of the risky asset. As in Back [3], based on the observation of risky asset's price dynamics, "the ordinary" agent can infer the additional information of the "informed" agent. The purpose of the study is to see whether an equilibrium condition can be attained by linear dynamics of an admissible price process of the risky asset for a given linear supply dynamics. Like in the Grossman-Stiglitz model, the market is Walsarian, i.e. the agents involved in the market are competitive agents.

Our studies show that the existence of linear competitive market equilibrium under asymmetric information is directly related to the existence of solution to some associated nonlinear equations. Indeed, the equilibrium condition can be explicitly expressed in the form of a system of nonlinear equations. However, we may not determine whether the associated system of nonlinear equations leads to a nonempty set of solution. We nevertheless find that in the particular case where the total supply is a Brownian motion, the equilibrium can be reached and we explicitly obtain the linear dynamics of an admissible price process.

The plan of the paper is organized as follows. We define the model and the equilibrium condition in section 5.2, while in section 5.3, we use stochastic control techniques and filtering theory to solve agents' CARA optimization problem and then determine their optimal trading portfolio. In section 5.4 and section 5.5, we express the characterization of a potential equilibrium price and explicitly calculate the linear dynamics of an admissible price process in the particular case where the total supply dynamics is a Brownian motion.

5.2 The model

We consider a financial market with a risky stock and a risk-free bond. The risk-free interest rate is assumed to be zero. We are given a standard Brownian motion, $W{=}(W_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions. T is a fixed time at which all transactions are liquidated.

5.2.1 Information and agents

There are two rational competitive traders:

- The first one is an "informed" trader (insider trader), agent I, whose information is described by the filtration, $\mathbb{F}$, as he can observe both the risky asset price $S = (S_t)$ and the total supply of the risky asset $Z = (Z_t)$. He has a Constant Absolute Risk Aversion (CARA) with coefficient $\eta_{_I} > 0$, i.e. his utility function is equal to $U_{_I}(v) = -\exp(-\eta_{_I} v)$.
- The second trader is an ordinary economic agent, agent O, whose information is only given by the price observation. We denote by $\mathbb{F}^O$ the structure of his filtration. He also has a Constant Absolute Risk-Aversion (CARA) with coefficient $\eta_{_O} > 0$, i.e. his utility function is in the form : $U_{_O}(v) = -\exp(-\eta_{_O} v)$.

We assume that the supply Z of the risky asset is a gaussian process, governed by the s.d.e:

$$dZ_t = (a(t)Z_t + b(t))\,dt + \gamma(t)dW_t, \quad Z_0 = z_0 \in \mathbb{R}, \tag{5.2.1}$$

where a, b, and γ are deterministic continuous functions from $[0,T]$ into $\mathbb{R}$.

5.2.2 Admissible price function

The purpose of this study is to find out whether an equilibrium condition can be attained by linear admissible price processes of the risky asset for a given linear supply dynamics as defined in (5.2.1). An admissible price process under $(\mathbb{P}, \mathbb{F})$ is a process in the form of :

$$dS_t = S_t \left[(\alpha(t) Z_t + \beta(t))\, dt + \sigma(t) dW_t \right], \quad 0 \leq t \leq T \tag{5.2.2}$$

where α and β are continuous functions from $[0, T]$ into $\mathbb{R}$, and σ a continuous function from $[0, T]$ into $\mathbb{R}_+^*$. We define $\mathcal{S}$ as the set of admissible price processes of risky asset.

The purpose is therefore to determine all set of functions (α, β, σ), i.e. admissible price processes, satisfying an equilibrium condition.

5.2.3 Equilibrium

Given an admissible price process S, a trading strategy for the "informed" agent (resp. the ordinary agent) is a $\mathbb{F}$ (resp. $\mathbb{F}^{O}$)-predictable process X integrable with respect to S. $X = (X_t)$ represents here the amount invested in the stocks at time t. We denote by $\mathcal{A}(\mathbb{F})$ (resp. $\mathcal{A}(\mathbb{F}^{O})$) this set of trading strategies, $X = (X_t)_{0 \leq t \leq T}$, which satisfy the integrability criteria:

$$\int_0^T |X_t|^2 dt < \infty, \quad \mathbb{P}\ a.s. \tag{5.2.3}$$

Each rational agent's goal, with its own filtration, is to maximize his expected utility from terminal wealth. We now formulate the definition of market equilibrium based on the law of supply-demand as described by a Rational Expectation Equilibrium of the Grossman and Stiglitz model.

Definition 5.2.1 *A market equilibrium is a pair $(\hat{X}^{I}, \hat{X}^{O})$ and an element $\hat{S} \in \mathcal{S}$ such that :*

(i) $\hat{X}^{I}$ is the solution of the insider agent's optimization problem :

$$\max_{X \in \mathcal{A}(\mathbb{F})} \mathbb{E} \left[U_{I} \left(v_{I} + \int_0^T X_t \frac{d\hat{S}_t}{\hat{S}_t} \right) \right],$$

where $v_{I} \in \mathbb{R}$ is the initial capital of the insider.
(ii) $\hat{X}^{O}$ is the solution of the ordinary agent's optimization problem :

$$\max_{X \in \mathcal{A}(\mathbb{F}^{O})} \mathbb{E} \left[U_{O} \left(v_{O} + \int_0^T X_t \frac{d\hat{S}_t}{\hat{S}_t} \right) \right],$$

where $v_{O} \in \mathbb{R}$ is the initial capital of the ordinary agent.
(iii) the market clearing conditions hold :

$$\hat{X}_t^{I} + \hat{X}_t^{O} = Z_t, \quad 0 \leq t \leq T.$$

If $(\hat{X}^{I}, \hat{X}^{O}, \hat{S})$ is a market equilibrium, then we say that $\hat{S}$ is an equilibrium pricing rule.

5.3 CARA utility maximization

In this section, we determine the optimal trading portfolio of the ordinary and insider agents.

5.3.1 "Informed" agent's optimization problem

Given an admissible price process S, the self-financed wealth process of the investor with a trading portfolio $X \in \mathcal{A}(\mathbb{F})$ has a dynamics given by :

$$\begin{aligned} dV_t &= X_t \frac{dS_t}{S_t} \\ &= X_t \left[\alpha(t) Z_t + \beta(t)\right] dt + X_t \sigma(t) dW_t. \end{aligned}$$

The investor with initial wealth v_I and constant risk aversion $\eta_I > 0$ has to solve the optimization problem :

$$\mathcal{J}_I(v_I) = \sup_{X \in \mathcal{A}(\mathbb{F})} \mathbb{E}\left[-\exp\left(-\eta_I V_T\right)\right]. \tag{5.3.1}$$

We consider the related dynamic optimization problem : for all $(t, v, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$J_I(t, v, z) = \sup_{X \in \mathcal{A}(\mathbb{F})} \mathbb{E}\left[-\exp\left(-\eta_I V_T\right) | V_t = v, Z_t = z\right], \tag{5.3.2}$$

so that

$$\mathcal{J}_I(v_I) = J_I(0, v_I, z_I).$$

The nonlinear dynamic programming equation associated to the stochastic control problem (5.3.2) is :

$$\frac{\partial J_I}{\partial t}(t, v, z) + \sup_{x \in \mathbb{R}} \mathcal{L}^x J_I(t, v, z) = 0, \tag{5.3.3}$$

together with the terminal condition $J_I(T, v, z) = -\exp(-\eta_I v)$. Here $\mathcal{L}^x$ is the second order linear differential operator associated to the diffusion (V, Z) for the constant control $X = x$:

$$\begin{aligned} \mathcal{L}^x J_I &= x[\alpha z + \beta] \frac{\partial J_I}{\partial v} + [az + b] \frac{\partial J_I}{\partial z} + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 J_I}{\partial v^2} \\ &\quad + x \sigma \gamma \frac{\partial^2 J_I}{\partial v \partial z} + \frac{1}{2} \gamma^2 \frac{\partial^2 J_I}{\partial z^2}. \end{aligned}$$

We make the logarithm transformation:

$$J_I(t, v, z) = -\exp[-\eta_I v - \phi(t, z)].$$

Then the Bellman equation (5.3.3) becomes:

$$\frac{\partial \phi}{\partial t} + \mathcal{L}_Z \phi + \sup_{x \in \mathbb{R}} \left[\eta_I x(\alpha z + \beta) - \frac{1}{2} \left|\eta_I x \sigma + \gamma \frac{\partial \phi}{\partial z}\right|^2\right] = 0, \tag{5.3.4}$$

together with the terminal condition :

$$\phi(T,z) \quad = \quad 0, \tag{5.3.5}$$

Here $\mathcal{L}_Z$ is the second order linear operator associated to the diffusion Z :

$$\mathcal{L}_Z\phi \quad = \quad (az+b)\frac{\partial\phi}{\partial z} + \frac{1}{2}\gamma^2\frac{\partial^2\phi}{\partial z^2}.$$

The maximum in (5.3.4) is attained for :

$$\hat{x}(t,z) \quad = \quad \frac{1}{\eta_I\sigma^2}\left[\alpha(t)z + \beta(t) - \sigma(t)\gamma(t)\frac{\partial\phi}{\partial z}(t,z)\right]. \tag{5.3.6}$$

Substituting into (5.3.4) gives :

$$\frac{\partial\phi}{\partial t} + \mathcal{L}_Z\phi + \frac{1}{2\sigma^2}\left(\alpha z + \beta - \sigma\gamma\frac{\partial\phi}{\partial z}\right)^2 - \frac{1}{2}(\gamma\frac{\partial\phi}{\partial z})^2 \quad = \quad 0. \tag{5.3.7}$$

This is a semi-linear equation for ϕ but with a quadratic term in $\frac{\partial\phi}{\partial z}$. We are therefore looking for a quadratic solution :

$$\phi(t,z) = \frac{1}{2}P_I(t)z^2 + Q_I(t)z + \chi_I(t)$$

where P_I, Q_I, and χ_I are deterministic functions valued in $\mathbb{R}$. By substituting and cancelling quadratic terms in z, we see that (5.3.7) holds iff P_I, Q_I and χ_I satisfy:

$$\begin{aligned}
0 &= \dot{P}_I + 2\left[a - \frac{\gamma\alpha}{\sigma}\right]P_I + \frac{\alpha^2}{\sigma^2} && (5.3.8)\\
P_I(T) &= 0, \\
0 &= \dot{Q}_I + \left[a - \frac{\gamma\alpha}{\sigma}\right]Q_I + \frac{\beta}{\sigma^2}\left[\alpha - \gamma\sigma P_I\right] + bP_I && (5.3.9)\\
Q_I(T) &= 0, \\
0 &= \dot{\chi}_I + \frac{\beta^2}{2\sigma^2} + \left[b - \frac{\gamma\beta}{\sigma}\right]Q_I + \frac{1}{2}\gamma^2P_I && (5.3.10)\\
\chi_I(T) &= 0.
\end{aligned}$$

By solving these differential equations, we obtain:

$$\begin{aligned}
P_I(t) &= \exp\left[2\int_t^T\left(a - \frac{\gamma\alpha}{\sigma}\right)(u)du\right] && (5.3.11)\\
&\quad \int_t^T\frac{\alpha^2}{\sigma^2}(s)\exp\left[-2\int_s^T\left(a - \frac{\gamma\alpha}{\sigma}\right)(u)du\right]ds, \\
Q_I(t) &= \exp\left[\int_t^T\left(a - \frac{\gamma\alpha}{\sigma}\right)(u)du\right] && (5.3.12)
\end{aligned}$$

$$\int_t^T \left[\frac{\beta}{\sigma^2}(\alpha - P_I\gamma\sigma) + P_I b\right](s)\exp\left[\int_s^T -\left(a - \frac{\gamma\alpha}{\sigma}\right)(u)du\right] ds,$$

$$\chi_I(t) = \int_t^T \left[\frac{\beta^2}{2\sigma^2} + (b - \frac{a\beta}{\sigma})Q_I + \frac{1}{2}\gamma^2 P_I\right](u)du. \tag{5.3.13}$$

The main result of this section can then be stated as follows:

Theorem 5.3.1 *The value function for problem* (5.3.2) *is equal to:*

$$J_I(t,v,z) = -\exp\left(-\eta_I v - \frac{1}{2}z^2 P_I(t) - Q_I(t)z - \chi_I(t)\right),$$

where P_I, Q_I and χ_I are expressed in (5.3.11), (5.3.12), *and* (5.3.13). *Moreover, the optimal trading portfolio for problem* (5.3.1) *is given by $\hat{X}_t^I = \hat{x}_I(t, Z_t)$, $0 \le t \le T$, where $\hat{x}_I(t,z)$ is defined on $[0,T]\times\mathbb{R}$ by :*

$$\hat{x}_I(t,z) = \Phi_I(t)z + H_I(t), \tag{5.3.14}$$

$$\Phi_I(t) = \frac{1}{\eta_I\sigma^2(t)}\left[\alpha(t) - \sigma(t)\gamma(t)P_I(t)\right], \tag{5.3.15}$$

$$H_I(t) = \frac{1}{\eta_I\sigma^2(t)}\left[\beta(t) - \sigma(t)\gamma(t)Q_I(t)\right]. \tag{5.3.16}$$

Proof. See Appendix 1. □

5.3.2 Ordinary agent's optimization problem

We now focus on the ordinary agent's optimization problem. To do so, we need to decompose the price process $(S_t)_t$ in its own filtration $\mathbb{F}^O = (\mathcal{F}_t^O)_{t\in[0,T]}$, which is generated by the price process, $\mathcal{F}_t^O = \sigma(S_s, s \le t)$.

We define:

$$\tilde{Z}_t = \mathbb{E}\left(Z_t | \mathcal{F}_t^O\right),$$

$$\Gamma(t) = \mathbb{E}\left[(Z_t - \tilde{Z}_t)^2\right].$$

From Kalman Bucy filter results (see Theorem 10.3 in [49]), $\tilde{Z}_t$ and Γ_t are solution of the system of equations:

$$\begin{cases} d\tilde{Z}_t = \left[a(t)\tilde{Z}_t + b(t)\right]dt + \frac{1}{\sigma(t)}\left[\sigma(t)\gamma(t) + \alpha(t)\Gamma(t)\right]dW_t^O, \\ \dot{\Gamma}(t) = 2a(t)\Gamma(t) - \frac{1}{\sigma^2(t)}\left[\gamma(t)\sigma(t) + \alpha(t)\Gamma(t)\right]^2 + \gamma^2(t), \end{cases} \tag{5.3.17}$$

where W^O a $(\mathbb{P}, \mathbb{F}^O)$-Brownian motion, the so-called innovation process.

We may obtain explicitly the expression of $\Gamma(t)$ by solving the Riccati equation (5.3.17) (See page 4-7 [60]):

$$\Gamma(t) \quad = \quad \Gamma(0)\frac{\exp\left(-\int_0^t\left[-2a(s)+2\frac{\gamma(s)\alpha(s)}{\sigma(s)}\right]ds\right)}{1+\Gamma(0)\int_0^t\frac{\alpha^2(s)}{\sigma^2(s)}\exp\left(-\int_0^s\left[-2a(u)+2\frac{\gamma(u)\alpha(u)}{\sigma(u)}\right]du\right)ds}.$$

The dynamics of an admissible price process under $(\mathbb{P},\mathbb{F}^O)$ is then given by :

$$dS_t \quad = \quad S_t\left[\left(\alpha(t)\tilde{Z}_t+\beta(t)\right)dt+\sigma(t)dW_t^O\right]. \tag{5.3.18}$$

The equivalent optimization problem for the ordinary agent with an initial wealth v_O and constant risk aversion $\eta_O > 0$ is :

$$\mathcal{J}_O(v_O) \quad = \quad \sup_{X\in\mathcal{A}(\mathbb{F}^O)}\mathbb{E}\left[-\exp\left(-\eta_O V_T\right)\right]. \tag{5.3.19}$$

We consider the related dynamic optimization problem : for all $(t,v,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$,

$$J_O(t,v,z) \quad = \quad \sup_{X\in A(\mathbb{F}^O)}\mathbb{E}\left[-\exp\left(-\eta_O V_T\right)\middle| V_t=v,Z_t=z\right], \tag{5.3.20}$$

so that $\mathcal{J}_O(v_O)=J_O(0,v_O,z_O)$.

Using the same arguments as in Theorem 5.3.1, we obtain the following results for ordinary agent:

Theorem 5.3.2 *The optimal trading portfolio for problem* (5.3.19) *is given by* $\hat{X}_t^O=\hat{x}_O(t,Z_t)$, $0\le t\le T$, *where* $\hat{x}_O(t,z)$ *is defined on* $[0,T]\times\mathbb{R}$ *by :*

$$\hat{x}_O(t,z) \quad = \quad \Phi_O(t)z+H_O(t), \tag{5.3.21}$$

$$\Phi_O(t) \quad = \quad \frac{1}{\eta_O\sigma^2(t)}\left[\alpha(t)-\sigma(t)\overline{\gamma}(t)P_O(t)\right], \tag{5.3.22}$$

$$H_O(t) \quad = \quad \frac{1}{\eta_O\sigma^2(t)}\left[\beta(t)-\sigma(t)\overline{\gamma}(t)Q_O(t)\right], \tag{5.3.23}$$

and P_O *and* Q_O *are expressed as:*

$$\begin{aligned}P_O(t) \quad &= \quad \exp\left[2\int_t^T\left(a-\frac{\overline{\gamma}\alpha}{\sigma}\right)(s)ds\right]\\&\int_t^T\frac{\alpha^2}{\sigma^2}\exp\left[-2\int_s^T\left(a-\frac{\overline{\gamma}\alpha}{\sigma}\right)(u)du\right](s)ds,\end{aligned} \tag{5.3.24}$$

$$\begin{aligned}Q_O(t) \quad &= \quad \exp\left[\int_t^T\left(a-\frac{\overline{\gamma}\alpha}{\sigma}\right)(s)ds\right]\int_t^T\left[\frac{\beta}{\sigma^2}\left[\alpha-P_O\overline{\gamma}\sigma\right]+P_Ob\right]\\&\exp\left[-\int_s^T\left(a-\frac{\overline{\gamma}\alpha}{\sigma}\right)(u)du\right](s)ds,\end{aligned} \tag{5.3.25}$$

with

$$\overline{\gamma} \quad = \quad \frac{1}{\sigma}[\sigma\gamma+\alpha\Gamma]. \tag{5.3.26}$$

5.4 Characterization of the equilibrium price

In this section, we give a characterization of a market equilibrium as defined in Definition 5.2.1. Using the optimal strategy of each agent determined in the previous section, we find that the equilibrium condition can be explicitly expressed as a nonlinear system.

Theorem 5.4.1 *The equilibrium condition is equivalent to the following nonlinear system of at most three equations with three unknown variables, α, β, and σ :*

$$\begin{cases} \frac{1}{\eta_O}\left[\beta(t) - \sigma(t)\overline{\gamma}(t)Q_O(t)\right] + \frac{1}{\eta_I}\left[\beta(t) - \sigma(t)\gamma(t)Q_I(t)\right] & = & 0, \\ \frac{1}{\eta_I\sigma^2(t)}\left[\alpha(t) - \sigma(t)\gamma(t)P_I(t)\right] - 1 & = & 0, \\ \left[\alpha(t) - \sigma(t)\overline{\gamma}(t)P_O(t)\right] Var(\tilde{Z}_t) & = & 0. \end{cases} \tag{5.4.1}$$

Proof. The equilibrium pricing rule is given by

$$\hat{X}_I(t, Z_t) + \hat{X}_O(t, \tilde{Z}_t) = Z_t. \tag{5.4.2}$$

To simplify the calculations, we assume w.l.o.g. that the gaussian process $(Z_t - \tilde{Z}_t, \tilde{Z}_t)$ is centered. The equilibrium (5.4.2) is equivalent to

$$\begin{cases} \mathbb{E}[\hat{X}_I(t, Z_t) + \hat{X}_O(t, \tilde{Z}_t)] & = & \mathbb{E}[Z_t], \\ \mathrm{Var}\left[\hat{X}_I(t, Z_t) + \hat{X}_O(t, \tilde{Z}_t)\right] & = & \mathrm{Var}(Z_t). \end{cases} \tag{5.4.3}$$

Using (5.3.14) and (5.3.21), the equilibrium condition becomes:

$$\begin{cases} H_O(t) + H_I(t) & = & 0, \\ (\Phi_I(t) - 1)\,\Gamma(t) & = & 0, \\ (\Phi_I(t) + \Phi_O(t) - 1)\,\mathrm{Var}(\tilde{Z}_t) & = & 0. \end{cases} \tag{5.4.4}$$

Since $\Gamma(t) > 0$, the above equilibrium condition is also written as:

$$\begin{cases} H_O(t) + H_I(t) & = & 0, \\ \Phi_I(t) - 1 & = & 0, \\ \Phi_O(t)\mathrm{Var}(\tilde{Z}_t) & = & 0. \end{cases} \tag{5.4.5}$$

and the required results are obtained by substituting the expression of H_O,H_I, Φ_O, and Φ_I. □

Remark 5.4.1 While the explicit expression of the equilibrium condition is in the form of a nonlinear system, we do not know whether this system leads to a nonempty set of solution. Recall that Q_O, Q_I, P_O, P_I, and $\overline{\gamma}$ are dependent on the unknown variables α, β, and σ, see (5.3.11), (5.3.12), (5.3.24), and (5.3.25).

Remark 5.4.2 In the case of a non-degenerated model, i.e. $\tilde{Z} \neq 0$, the equilibrium is equivalent to the following system:

$$\begin{cases} \frac{1}{\eta_O}\left[\beta(t) - \sigma(t)\overline{\gamma}(t)Q_O(t)\right] + \frac{1}{\eta_I}\left[\beta(t) - \sigma(t)\gamma(t)Q_I(t)\right] & = & 0, \\ \frac{1}{\eta_I\sigma^2(t)}\left[\alpha(t) - \sigma(t)\gamma(t)P_I(t)\right] - 1 & = & 0, \\ \alpha(t) - \sigma(t)\overline{\gamma}(t)P_O(t) & = & 0. \end{cases}$$

5.5 Equilibrium in the case: $Z_t = W_t$

We take the particular case of $Z_t = W_t$, i.e. $a(t) = b(t) = 0$ and $\gamma(t) = 1$.

Proposition 5.5.1 *In the case of $Z_t = W_t$, the equilibrium is reached and the linear dynamics of an admissible price process is given by*

$$dS_t = S_t\left[\alpha(t)Z_t dt + \sigma(t)dW_t\right]. \tag{5.5.6}$$

with

$$\sigma(t) = \frac{1}{\eta_I}\left[\mu(t) + \frac{1}{3}\frac{\mu^2(t)}{\mu_T}\left(1 - \frac{\mu^3(t)}{\mu_T^3}\right)\right], \tag{5.5.7}$$

$$\alpha(t) = \sigma(t)\mu(t), \tag{5.5.8}$$

where

$$\mu(t) = \frac{\mu_T}{1 + \mu_T(T-t)}$$ *and μ_T is any arbitrary positive constant.*

Remark 5.5.3 The equilibrium condition does not depend on the CARA coefficient of the ordinary agent. In economic sense, this means that the "informed" agent defines his trading strategy in order to maximise his expected utility from terminal wealth and imposes his optimal trading strategy upon the ordinary trader.

Proof of proposition 5.5.1. Let us set $\mu(t) = \frac{\alpha(t)}{\sigma(t)}$. From (5.3.26), (5.3.17), and (5.3.8), we obtain :

$$\begin{cases} \overline{\gamma}(t) = 1 + \mu(t)\Gamma(t), \\ \dot{\Gamma}(t) = 1 - [1 + \mu(t)\Gamma(t)]^2, \\ \dot{P}_O = -\mu(t)^2 + 2\mu(t)[1 + \mu(t)\Gamma(t)]P_O(t). \end{cases} \tag{5.5.9}$$

While the first relation in (5.3.17) becomes:

$$d\tilde{Z}_t = [1 + \mu(t)\Gamma(t)]dW_t^O. \tag{5.5.10}$$

Thus

$$\mathbf{Var}(\tilde{Z}_t) = \int_0^t [1 + \mu(s)\Gamma(s)]^2 ds. \tag{5.5.11}$$

The equilibrium pricing rule (5.4.1) becomes :

$$\frac{1}{\eta_O}\left[\beta(t) - \sigma(t)(1 + \mu(t)\Gamma(t))Q_O(t)\right] + \frac{1}{\eta_I}\left[\beta(t) - \sigma(t)Q_I(t)\right] = 0, \tag{5.5.12}$$

$$\frac{1}{\eta_I\sigma(t)}\left[\mu(t) - P_I(t)\right] - 1 = 0, \tag{5.5.13}$$

$$\left[\mu(t) - (1 + \mu(t)\Gamma(t))P_O(t)\right]\mathrm{Var}(\tilde{Z}_t) = 0. \tag{5.5.14}$$

The latter relation (5.5.14) is equivalent to, for all $t \in [0,T]$:

$$\mu(t) - (1+\mu(t)\Gamma(t))P_O(t) \;=\; 0, \tag{5.5.15}$$

or

$$\mathrm{Var}(\tilde{Z}_t) \;=\; 0, \quad \forall\, s \in\; [0,t]. \tag{5.5.16}$$

We show that the degenerated case (5.5.16) cannot happen. Assume that there exists t such that the latter equation (5.5.16) is satisfied, then by using (5.5.11), we have

$$1+\mu(s)\Gamma(s) \;=\; 0,\ \forall\, s \in\; [0,t].$$

As $\Gamma_s = s$, we obtain:

$$\mu(s) = -\frac{1}{s}, \quad \forall\, s \;\in\; [0,t]. \tag{5.5.17}$$

We recall that $\mu = \frac{\alpha}{\beta}$, as such, a straight calculation gives us the expression of α and σ in the interval $[0,t]$, whose values would explode at time 0, leading to non admissible price function.

As such, relation (5.5.14) is equivalent to

$$\mu(t) - (1+\mu(t)\Gamma(t))P_O(t) \;=\; 0, \tag{5.5.18}$$

By deriving any of the latter equation and using the expressions of $\dot{\Gamma}$ and $\dot{P}_O$ in (5.5.9) , we obtain the following equation for μ:

$$\frac{\dot{\mu}(t)}{\mu^2(t)} \;=\; 1, \quad t \in [0,T] \tag{5.5.19}$$

As such,

$$\mu(t) \;=\; \mu_T \frac{1}{1+\mu_T(T-t)} \tag{5.5.20}$$

which raises no problem of definition in the case of $\mu_T \;>\; 0$.
From equation (5.5.12) and (5.5.13),we obtain the explicit expression of σ and β, and therefore α.

$$\begin{aligned}
\sigma(t) &= \frac{1}{\eta_I}\left[\mu(t) + \frac{1}{3}\frac{\mu^2(t)}{\mu_T}\left(1 - \frac{\mu^3(t)}{\mu_T^3}\right)\right] && (5.5.21)\\
\alpha(t) &= \sigma(t)\mu(t) && (5.5.22)\\
\beta(t) &= 0 && (5.5.23)\\
\text{where}\quad \mu(t) &= \frac{\mu_T}{1+\mu_T(T-t)}. && (5.5.24)
\end{aligned}$$

We check that when $\mu_T > 0$, μ and σ are positive for $t \in [0,T]$. □

Appendix: Proof of Theorem 5.3.1

We set:

$$\begin{aligned}\phi(t,z) &= \frac{1}{2}P_I(t)z^2 + Q_I(t)z + \chi_I(t)\\ g(t,v,z) &= -\eta_I v - \phi(t,z)\end{aligned}$$

Where P_I, Q_I, and χ_I are expressed as above $[see(5.3.11),(5.3.12),(5.3.13)]$
By differentiating, we obtain:

$$\frac{\partial g}{\partial t} = -\frac{\partial \phi}{\partial t}, \frac{\partial g}{\partial v} = -\eta_I, \frac{\partial g}{\partial z} = -\frac{\partial \phi}{\partial z}$$
$$\mathcal{L}^x g = -\eta_I(\alpha z + \beta)x - \mathcal{L}_Z\phi$$

By applying Itô's formula to $g(t,V_t,Z_t)$ for any $X \in \mathcal{A}(\mathbb{F})$ between t and T, we obtain :

$$\begin{aligned}g(T,V_T,Z_T) &= g(t,V_t,Z_t) + \int_t^T \left(\frac{\partial g}{\partial t} + \mathcal{L}^{X_u} g\right)(u,V_u,Z_u)du\\ &\quad + \int_t^T (\frac{\partial g}{\partial v}X\sigma + (\frac{\partial g}{\partial z})\gamma)(u,V_u,Z_u)dB_u\\ &= g(t,V_t,Z_t) + \int_t^T \left(-\frac{\partial \phi}{\partial t} - \mathcal{L}_Y\phi - \eta_I X(\alpha Z+\beta)\right)(u,Z_u)du\\ &\quad + \int_t^T \left(-\eta_I X_u\sigma(u) - (\frac{\partial \phi}{\partial z})(u,Z_u)\gamma(u)\right)dB_u\\ &= g(t,V_t,Z_t) - \int_t^T \left(\frac{\partial \phi}{\partial t} + \mathcal{L}_Y\phi + \eta_I X_u(\alpha Z+\beta)\right.\\ &\quad \left. -\frac{1}{2}\left|\eta_I X\sigma + \gamma\frac{\partial \phi}{\partial z}\right|^2\right)(u,Z_u)du\\ &\quad - \int_t^T \left(\eta_I X_u\sigma(u) + (\frac{\partial \phi}{\partial z})(u,Z_u)\gamma(u)\right)dB_u\\ &\quad -\frac{1}{2}\int_t^T \left|\eta_I X\sigma + \gamma\frac{\partial \phi}{\partial z}\right|^2 (u,Z_u)du. \end{aligned} \tag{5.A.1}$$

We now consider the exponential local $(\mathbb{P},\mathbb{F})$-martingale for any $X \in \mathcal{A}(\mathbb{F})$:

$$\begin{aligned}\xi_t^X &= \exp\left\{-\int_t^T \left(\eta_I X_u\sigma(u) + (\frac{\partial \phi}{\partial z})(u,Z_u)\gamma(u)\right)dB_u\right.\\ &\quad \left. -\frac{1}{2}\int_t^T \left|\eta_I X\sigma + \gamma\frac{\partial \phi}{\partial z}\right|^2 (u,Z_u)du.\right\}.\end{aligned}$$

From PDE (5.3.4) satisfied by ϕ, relation (5.A.1) yields for all $X \in \mathcal{A}(\mathbb{F})$:

$$\exp\left(g(T,V_t,Z_T)\right) \;\geq\; \exp\left(g(t,V_t,Z_t)\right).\frac{\xi_T^X}{\xi_t^X}. \tag{5.A.2}$$

Since $g(T,v,z) = -\eta_{_I} v$ and ξ^X is a $(\mathbb{P},\mathbb{F})$-supermartingale, we obtain by taking conditional expectation in the previous inequality :

$$\mathbb{E}\left[-\exp\left(-\eta_{_I} V_T\right)\middle| V_t = v, Y_t = y\right] \quad \leq \quad -\exp\left(g(t,v,z)\right),$$

for all $X \in \mathcal{A}(\mathbb{F})$ and so :

$$J_{\mathbb{F}}(t,v,z) \quad \leq \quad -\exp\left(g(t,v,z)\right). \tag{5.A.3}$$

Consider now the control strategy $\hat{X}_t = \hat{x}(t,Z_t)$, $0 \leq t \leq T$, where $\hat{x}$ is defined in (5.3.6) or more explicitly in (5.3.14). Then, we clearly have $\hat{X} \in \mathcal{A}(\mathbb{F})$, and we have now equality in (5.A.2) since $\hat{x}$ attains the supremum in the PDE (5.3.4) :

$$\exp\left(g(T,V_t,Z_T)\right) \quad = \quad \exp\left(g(t,V_t,Z_t)\right).\frac{\xi_T^{\hat{X}}}{\xi_t^{\hat{X}}}. \tag{5.A.4}$$

Observe that :

$$\begin{aligned}\eta_{_I}\hat{X}_u\sigma(u) + (\frac{\partial\phi}{\partial z})(u,Z_u)\gamma(u) \quad &= \quad Z_u(\eta_{_I}\Phi(u)\sigma(u) + P_{_I}(u)\gamma(u)) + \\ &\qquad \eta_{_I} H(u)\sigma(u) + Q_{_I}(u)\gamma(u).\end{aligned}$$

Since Z is a Gaussian process, it follows that for some $\delta > 0$, we have :

$$\mathbb{E}\left[\exp\left(\delta\left|\eta_{_I} X_u\sigma + \gamma\frac{\partial\phi}{\partial z}\right|^2(u,Z_u)\right)\right] \quad < \quad \infty.$$

Therefore by Lipster, Shiryaev (1977, p.220), $\xi^{\hat{X}}$ is a $(\mathbb{P},\mathbb{F})$-martingale and so by taking conditional expectation in (5.A.4), we have :

$$\mathbb{E}\left[-\exp\left(-\eta_{_I} V_T\right)\middle| V_t = v, Z_t = z\right] \quad = \quad -\exp\left(g(t,v,z)\right),$$

for the wealth process V controlled by the trading portfolio $\hat{X}$. This last equality combined with (5.A.3) ends the proof. □

Bibliography

[1] Akian M., Sulem A. and M. Taksar (2001) : "Dynamic optimization of long term growth rate for a portfolio with transaction costs and logarithm utility", *Math. Finance*, **11**, 153-188.

[2] Almgren R. and N. Chriss (2001) : "Optimal execution of portfolio transactions", *Journal of Risk*, **3**, 5-39.

[3] Back K. (1992) : "Insider trading in continuous time", *Review of Financial Studies*, **5**, 387-409.

[4] Bank P. and D. Baum (2004) : "Hedging and portfolio optimization in illiquid financial markets with a large trader", *Mathematical Finance*, **14**, 1-18.

[5] Barles G. (1994) : Solutions de viscosité des équations d'Hamilton-Jacobi, *Math. et Appli.*, Springer Verlag.

[6] Barles G. and P.E. Souganidis (1991) : "Convergence of approximation schemes for fully nonlinear second order equations",*Asymptotic analysis*, **4**, 271-283.

[7] Bensoussan A. and J.L. Lions (1982) : Impulse control and quasi-variational inequalities, Gauthiers-Villars.

[8] Biais B. and J.C. Rochet (1997) : Risk Sharing, adverse selection and market structure, Financial Mathematics, Lectures Notes on Mathematics, Springer Verlag.

[9] Boguslavskaya E.: "On Optimization of dividend flow for a company in a presence of liquidation value", Working paper. http://www.boguslavsky.net/fin/index.html

[10] Bouchard B., I. Ekeland and N. Touzi (2002). "On the Malliavin approach to Monte Carlo approximation of conditional expectations", *Finance and Stochastics*, Vol **8**, 45-72, Springer-Verlag 2004

[11] Bouchard B. and N. Touzi (2004) : "Discrete time approximation and Monte Carlo simulation of Backward Stochastic Differential Equations", *Stochastic Processes and their applications*, Vol **111**,175-206.

[12] Brekke K. and B. Oksendal (1994) : "Optimal switching in an economic activity under uncertainty", *SIAM J. Cont. Optim.*, 32, 1021-1036.

[13] Cadenillas A. and F. Zapatero (1999) : "Optimal central bank intervention in the foreign exchange market", *Journal of Econ. Theory*, **97**, 218-242.

[14] Cetin U., Jarrow R. and P. Protter (2004) : "Liquidity risk and arbitrage pricing theory", *Finance and Stochastics*, **8**, 311-341.

[15] Cetin U. and C. Rogers (2005) : "Modelling liquidity effects in discrete-time", to appear in *Mathematical Finance.*

[16] Cetin U., Soner M. and N. Touzi (2006) : "Options hedging under liquidity costs", working paper.

[17] Chancelier J.P., Oksendal B. and A. Sulem (2001) : "Combined stochastic control and optimal stopping, and application to numerical approximation of combined stochastic and impulse control", *Stochastic Financial Mathematics*, Proc. Steklov Math. Inst. Moscou, 149-175, ed. A. Shiryaev.

[18] Choulli T. , Taksar M. and X.Y. Zhou (2003) : "A diffusion model for optimal dividend distribution for a company with constraints on risk control", *SIAM J. Cont. Optim.*, **41**, 1946-1979.

[19] Crandall M., Ishii H. and P.L. Lions (1992) : "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.*, **27**, 1-67.

[20] Cuoco D. and J. Cvitanic (1998) : "Optimal consumption choice for a large investor", *Journal of Economic Dynamics and Control*, **22**, 401-436.

[21] Cvitanic J. and I. Karatzas (1996) : "Hedging and portfolio optimization under transaction costs : a martingale approach", *Mathematical Finance*, 6, 133-165.

[22] Davis M. and A. Norman (1990) : "Portfolio selection with transaction costs", *Math. of Oper. Research*, **15**, 676-713.

[23] Davis M.H.A. and M. Zervos (1994) : "A Problem of Singular Stochastic Control with Discretionary Stopping", *Annals of Applied Probability*, **4**, pp.226-240.

[24] Décamps J.P. and S. Villeneuve (2005) : "Optimal dividend policy and growth option", preprint GREMAQ, Toulouse University.

[25] Deelstra G., Pham H. and N. Touzi (2001) : "Dual formulation of the utility maximization problem under transaction costs", *Annals of Applied Probability*, **11**, 1353-1383.

[26] Dixit A. and R. Pindick (1994) : Investment under uncertainty, Princeton University Press.

[27] Duckworth K. and M. Zervos (2001) : "A model for investment decisions with switching costs", *Annals of Applied Probability*, 11, 239-250.

[28] Fleming W. and M. Soner (1993) : Controlled Markov processes and viscosity solutions, Springer Verlag, New York.

[29] Fournié E., J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi(1999). Applications of Malliavin calculus to Monte Carlo methods.

[30] Frey R. (1998) : "Perfect option hedging for a large trader", *Finance and Stochastics*, **2**, 115-141.

[31] Gilbarg D. and N. Trudinger (1977) : Elliptic partial differential equations of second order, Springer Verlag, Berlin.

[32] Glosten L.R. and P.R. Milgrom (1985): "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders". *Journal of Financial Economics* **14**, 71-100.

[33] Grorud, A. and M. Pontier (1998): "Insider trading in a continuous time market model". *International Journal of Theoritical and Applied Finance* **1**, 331-347.

[34] Grossman S.J. and J.E. Stiglitz (1980): "On the Impossibility of Informationally Efficient Markets". *American Economic Review* **70**, 393-408.

[35] Guo X. (2001) : "An explicit solution to an optimal stopping problem with regime switching", *Journal of Applied Probability*, 38, 464-481.

[36] Guo X. and H. Pham (2005) : "Optimal partially reversible investment with entry decision and general production function", *Stochastic Processes and their Applications*, **115**, 705-736.

[37] Hamadène S. and M. Jeanblanc (2005) : "On the Starting and Stopping Problem: Application in reversible investments", to appear in *Mathematics of Operation Research*.

[38] He H. and H. Mamaysky (2005) : "Dynamic trading policies with price impact", *Journal of Economic Dynamics and Control*, **29**, 891-930.

[39] Hillairet, C. (2004): "Existence of an equilibrium on a financial market with different assymetric information", *preprint* , LSP.

[40] Ishii K. (1993) : "Viscosity solutions of nonlinear second order elliptic PDEs associated with impluse control problems", *Funkcial. Ekvac.*, **36**, 123-141.

[41] Ishii H. and P.L. Lions (1990) : "Viscosity solutions of fully nonlinear second order elliptic partial differential equations", *Journal of Differential equations*, **83**, 26-78.

[42] Jacod J. (1985) : "Grossissement Initial, Hypothèse (H') et Théorème de Girsanov", *Lecture Notes* 118, Springer-Verlag, 15-35.

[43] Jeanblanc M. and A. Shiryaev (1995) : "Optimization of the flow of dividends", *Russian Math Surveys*, **50**, 257-277.

[44] Jeulin T. (1980) : "Semi-martingales et grossissement d'une filtration", *Lecture Notes Math* 833, Springer-Verlag.

[45] Jouini E. and H. Kallal (1995) : "Martingale and arbitrage in securities markets with transaction costs", *Journal of Econ. Theory*, **66**, 178-197.

[46] Karatzas I., D. Ocone, H. Wang and M. Zervos (2000) : "Finite-Fuel Singular Control with Discretionary Stopping", *Stochastics and Stochastics Reports*, **71**, pp.1-50

[47] Korn R. (1998) : "Portfolio optimization with strictly positive transaction costs and impulse control", *Finance and Stochastics*, **2**, 85-114.

[48] Kyle A. (1985) : "Continuous auctions and insider trading", *Econometrica*, **53**, 1315-1335.

[49] Lipster, R. Sh., Shiryaev, A. N. (2001), Statistics of Random Processes. I, New York, Springer-Verlag, 396.

[50] Lo A., Mamayski H. and J. Wang (2004) : "Asset prices and trading volume under fixed transcation costs", *J. Political Economy*, **112**, 1054-1090.

[51] Ly Vath V. and H. Pham (2005) : "Explicit solution to an optimal switching problem in the two-regime case", to appear in *SIAM Journal on Control and Optimization*.

[52] Ly Vath V., M. Mnif and H Pham (2005) : "A model of optimal portfolio selection under liquidity risk and price impact", to appear in *Finance and Stochastics*.

[53] Merton, R.C. (1971) : "Optimum consumption and portfolio rules in a continuous-time model", *J. Econ. Theory*, **3**, 373-413.

[54] Morton A. and S. Pliska (1995) : "Optimal portfolio management with fixed transaction costs", *Mathematical Finance*, **5**, 337-356.

[55] Oksendal B. and A. Sulem (2002) : "Optimal consumption and portfolio with both fixed and proportional transaction costs", *SIAM J. Cont. Optim.*, **40**, 1765-1790.

[56] Papanicolaou G. and R. Sircar (1998) : "General Black-Scholes models accounting for increased market volatility from hedging strategies", *Applied Mathematical Finance*, **5**, 45-82.

[57] Pham H. (2005) : "On the smooth-fit property for one-dimensional optimal switching problem", to appear in *Séminaire de Probabilités*, Vol. XL.

[58] Platen E. and M. Schweizer (1998) : "On feedback effects from hedging derivatives", *Mathematical Finance*, **8**, 67-84.

[59] Radner R. and L. Shepp (1996) : "Risk vs. profit potential : a model of corporate strategy", *Journal of economic dynamics and Control*, **20**, 1373-1393.

[60] Reid, W. (1972), Riccati differential equations, volume 86 of Mathematics in Science and Engineering, New York and London, Academic Press.

[61] Shreve S., Lehoczky J.P. and D. Gaver (1984) : "Optimal consumption for general diffusions with absorbing and reflecting barriers", *SIAM J. Cont. Optim.* , **22**, 55-75.

[62] Soner H. (1986) : "Optimal control with state-space constraint, I and II", *SIAM J. Cont. Optim.*, **24**, 552-561, and 1110-1122.

[63] Subramanian A. and R. Jarrow (2001) : "The liquidity discount", *Mathematical Finance*, **11**, 447-474.

[64] Tang S. and J. Yong (1993) : "Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach", *Stoch. and Stoch. Reports*, **45**, 145-176.

[65] Vayanos D. (1998) : "Transaction costs and asset prices : a dynamic equilibrium model", *Rev. Fin. Studies*, **11**, 1-58.

[66] Villeneuve S. (2005) : "On the threshold strategies and smooth-fit principle for optimal stopping problems", preprint GREMAQ Toulouse University.

[67] Zariphopoulou T. (1988) : Optimal investment-consumption models with constraints, PhD Thesis, Brown University.

[68] Zervos M. (2003) : "A Problem of Sequential Entry and Exit Decisions Combined with Discretionary Stopping", *SIAM Journal on Control and Optimization*, **42**, 397-421.

Résumé : Nous étudions quelques applications du contrôle stochastique aux options réelles et au risque de liquidité. Plus précisément, dans la première partie, nous nous intéressons à un problème de sélection du portefeuille optimal sous un modèle de risque de liquidité, puis dans la deuxième partie, à deux options réelles: un problème de changement de régime et un problème couplé de contrôle singulier et de changement de régime pour une politique de dividende avec investissement réversible, et enfin, dans la dernière partie, à l'existence d'un équilibre dans un marché compétitif sous asymétrie d'information. Dans la résolution de ces problèmes, surtout dans les deux premières parties, des techniques de contrôle stochastique seront utilisées. L'approche typique consiste à exprimer le principe de la programmation dynamique lié à chaque problématique afin d'obtenir une caractérisation par EDP des fonctions de valeur. Par cette approche, nous montrons, dans le problème de risque de liquidité et les deux options réelles, que les fonctions de valeur correspondantes sont l'unique solution du système d'inégalités variationnelles d'HJB associé. Dans chaque problème des deux premières parties, on peut obtenir les solutions, en particulier les contrôles optimaux, soit d'une manière explicite, soit par une méthode itérative.

Mots-clés : sélection de portefeuille, contrôle impulsionnel, risque de liquidité, contrôle singulier, changement de régime optimal, solution de viscosité, inégalités variationnelles, principe de "smooth-fit", information asymétrique, équilibre, théorie du filtrage.

Discipline : Mathématiques

Abstract : We study stochastic control applications to real options and to liquidity risk model. More precisely, we investigate, in the first part, a model of optimal portfolio selection under liquidity risk and price impact, then, in the second part, two real option problems: an optimal switching problem and a mixed singular/switching control problem for a dividend policy with reversible investment, and finally, in the third part, a competitive market equilibrium problem under asymmetric information. In the resolution of these problems, stochastic control techniques will be intensively used. The typical approach consists in expressing the dynamic programming principle related to each case, in order to obtain a PDE characterization of the value functions. Based on this approach, we show, in the liquidity risk problem and both real options, that the corresponding value functions are unique solution to the associated system of HJB variational inequalities. In each problem of the first two parts, we obtain the solutions, in particular the optimal control, either explicitly or via an iterative method.

Key words : portfolio selection, impulse control, liquidity risk, mixed singular/switching control problem, viscosity solution, optimal switching, variational inequalities, smooth-fit principle, asymmetric information , stochastic filtering theory, equilibrium.

Laboratoire de Probabilités et Modèles Aléatoires,
CNRS-UMR 7599, UFR de Mathématiques, case 7012
Université Paris 7, Denis Diderot
2, place Jussieu, 75251 Paris Cedex 05.